Physics AP in Review

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What with chromodynamics and electroweak too
Our Standardized Model should please even you,
Tho' once you did say that of charm there was none
It took courage to switch as to say Earth moves not Sun.
Yet your state of the union penultimate large
Is the last known haunt of the Fractional Charge,
And as you surf in the hot tub with sourdough roll
Please ponder the passing of your sole Monopole.
Your Olympics were fun, you should bring them all back
For transsexual tennis or Anamalon Track,
But Hollywood movies remain sinfully crude
Whether seen on the telly or Remotely Viewed.
Now fasten your sunbelts, for you've done it once more,
You said it in Leipzig of the thing we adore,
That you've built an incredible crystalline sphere
Whose German attendants spread trembling and fear
Of the death of our theory by Particle Zeta
Which I'll bet is not there say your article, later.
– SHELDON GLASHOW, *Physics Today*, December, 1984

Introduction

The purpose of this Physics Review is to cover, without redundancy or unnecessary elaboration, the important skills taught in Physics AP and applicable to the analysis and solution of complex problems that may appear on the AP Physics Mechanics and Electricity & Magnetism exams. Its main goal is to establish each aspect of AP-level problem solving physics into a cohesive, logical construct, encouraging the development of an interdependent body of knowledge which demands a solid understanding without gaps.

In other words, read this and figure out what parts you don't understand from other sources.

This review is not concerned with the history of physics, lists of constants, or descriptions of basic concepts of measurement.

Notation Semantics

There are a few organization and notation semantics used throughout this document:

Margin notes specify extra information, or humor.

$\equiv$ is the equivalence symbol; it can be read as "is defined as." It denotes mathematical definitions rather than derived relationships.

Definitions follow this format.

Mathematical expressions inline with text are used primarily for explanation, as are some equations displayed alone. However, most equations that are displayed by themselves are conclusions, and those that are numbered are recommended for memorization.[1]

This document is formatted with LaTeX and LyX. (http://www.lyx.org)

License

Requests

Although it is certainly not required, it would be really swell if you emailed me *(shimon@rura.org)* if you did anything more than skim this document and decide that it was worthless to you.

The author encourages criticisms, suggestions, alterations, translations, transformations, explosions, and cold fusion. Any correspondence related to these things should also be sent to the email address above.

[1] Footnotes elaborate or define things, or point you to other information.

Revision History

Beta (β) versions are incomplete, although believed to be correct, spell-checked, and otherwise appropriate for use. Alpha (α) versions contain material which has not been proofread and are for developers (writers?) only.

First Public Release, Nov 1, 2006
Updated contact information and posted on the web.

Version 1.0**, May 17, 1999**
Equation-completing chapters on Sources of the Magnetic Field, Faraday's Law, and Inductance; this will be all before the AP Exam. Equation-only extract will follow. 16 014 words long!

Version $\beta 6\,\alpha 1$**, March 22, 1999**
Added chapters on Capacitance, Current & Resistance, and Direct Current Circuits

Version $\beta 5\,\alpha 1$**, February 24, 1999**
Updated license to be truly free. Adding chapter on Gauss's Law. I am hoping someone else will rewrite *The Law of Universal Gravitation* (hence the α).

Version $\beta 4\,\alpha 1$**, February 8, 1999**
Switched to **book** textclass. Adding *Part II, Electricity and Magnetism*. *The Law of Universal Gravitation* still incomplete.

Version $\beta 3\,\alpha 1$**, January 22, 1999**
Added *Oscillatory Motion* and *The Law of Universal Gravitation.* Not totally complete.

Version $\beta 2$**, January 4, 1999**
Added *Static Equilibrium and Elasticity*.

Version $\beta 1$**, January 4, 1999**
First beta release. Contains sections through *Rotational Motion About a Non-fixed Axis.*

Contents

Part I

Mechanics

Chapter 1

Mathematical Foundations

The mathematical abilities necessary for the analysis of problems in AP Physics include arithmetic, algebra, trigonometry, and calculus. In addition, various mathematical skills pertinent primarily to physics, such as vectors, will be covered in this review.

1.1 Trigonometry

Right-triangle trigonometry is vital for the analysis of quantities and their components in physics. It is *very* important. You should have a thorough knowledge of the three trigonometric functions as well as their inverses, and especially their applicability in problem-solving.

1.2 Calculus

Often, it is necessary to calculate instantaneous quantities, or to incorporate a function into a calculation (rather than an average value). This is made possible using differentiation and integration.

1.2.1 Differentiation (the derivative)

The derivate of $f(x)$ is written $f'(x)$ or $\frac{d}{dx}f(x)$. Qualitatively, it is the slope of the line tangent to the curve $f(x)$ at a point. You should know how to calculate derivatives of functions and use them in solutions.

1.2.2 Integration (the integral or antiderivative)

The integral of $f(x)$ is written $\int f(x)$. Qualitatively, it is the area under the curve over a specified range. You should know how to calculate integrals and use them in solutions.

1.3 Vectors

A vector represents the magnitude of a quantity as well as its direction (whereas a scalar quantity only represents magnitude). Vectors can be represented in many forms, including positive and negative numbers, coordinate pairs, or legs of a right triangle.

1.3.1 Vector Components

In physics, it is often convenient to treat a situation in terms of the plane in which it rests (or moves), such as the Cartesian axes for two-dimensional motion. In this case, vectors that occur at oblique angles can be described as having *components* in the x- and y-directions. Using these components, which are determined by treating the vector as the hypotenuse of a right triangle and finding the legs, one can perform precise arithmetic with little difficulty: adding or subtracting vectors can be done by taking the sum or difference of their components in the dimensional directions.

1.3.2 Unit Vectors

An easy way of denoting vectors as the sum of their component vectors is *unit vector notation*. The unit vectors **i, j,** and **k** represent vectors of length 1 unit along the **x, y,** and **z** axes, respectively. So, if you have a vector A with x, y, and z components A_x, A_y, and A_z, you can write it in unit vector notation as $A = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$.

$$|i| = |j| = |k| = 1$$

Chapter 2

One-dimensional Motion

Kinematics is the study of motion using the concepts of space and time, without regard for the causes of motion.

Dynamics is concerned with the causes of motion (forces) and their effects upon moving objects.

In most physics problems involving translational motion, a moving body is treated as a single particle. This is usually a reasonable approximation.

"Consider a spherical bear, in simple harmonic motion..." – Professor in the UCB physics department

2.1 Position

Position, the most horrifyingly obvious part of kinematics, describes the location of a particle. As a function of time, it is often written $s(t)$ or $x(t)$.

2.2 Velocity

Velocity is the vector (the scalar analogue is speed) which describes the rate of change in an object's position. Thus:

$$v = \frac{ds}{dt} = \int a \tag{2.1}$$

Average velocity can be found by dividing displacement by time ($\frac{\Delta d}{\Delta t}$). Instantaneous velocity is found using the calculus in equation 2.1.

2.3 Acceleration

Acceleration is a vector (the scalar analogue, also acceleration, is added or subtracted) which describes the rate of change in an object's velocity. Its units are meters per second per second (or $\frac{m}{s^2}$). Thus:

$$a = \frac{dv}{dt} \tag{2.2}$$

Average acceleration can be found by dividing change in velocity by time ($\frac{\Delta v}{\Delta t}$). Instantaneous velocity is found using the derivative in equation 2.2.

2.4 Useful Kinematics Equations

These should probably be embedded into your equation-soaked brain by now. It's probably enough if they just make sense.

You should be fully able to derive these, but they are useful to memorize because they can be easily used to analyze each aspect of kinematics in a time crunch. Keep in mind, these work only with constants and give averages, and stuff like that.

$$v = v_0 + at \tag{2.3}$$

$$x - x_0 = \frac{1}{2}(v + v_0)t \tag{2.4}$$

$$x - x_0 = v_0 t + \frac{1}{2}at^2 \tag{2.5}$$

$$v = \sqrt{v_o^2 + 2a(x - x_0)} \tag{2.6}$$

2.5 Gravity

You may have noticed (of course, you'd have to be one of the observant ones) that, when you let go of something you're holding, it falls down. This is due to a magical, mystical, physical force called gravity which, among other things, keeps stuff like the atmosphere, your house, the moon, and you from floating away. When a body is in freefall, it is subjected to an acceleration due to gravity, usually represented as the constant g. So, when there's something moving along the same axis as gravity, you have to count gravity in the acceleration calculations. Capisce?

On earth, $g = 9.8\frac{m}{s^2}$.

Chapter 3

Two-dimensional Motion

So, you've got a particle moving in a plane. It's really the same as with motion in one dimension, except you're working with a vector.

3.1 The Displacement Vector

Often, the vector describing the position or displacement of a particle is labeled **r**. Just like with one dimensional motion, $v = \frac{\Delta r}{\Delta t}$, $a = \frac{\Delta v}{\Delta t}$.

3.2 Analyzing Planar Motion Problems

Analyzing the motion of a particle in two dimensions is just like motion in one dimension, simply done twice. After you've determined how it moves left and right, figure out how it moves up and down.

The important point here is that, when working out a problem, you have to take into account every initial condition and every force acting upon an object. The best way to do this in problems is by drawing diagrams which show initial conditions and forces acting upon the particle. This also provides an easy way to check if your answer makes sense.

For homework, draw 3000 diagrams for potential physics problems.

Vector math note: if you're looking for the magnitude of a vector (or, if you're speaking polar coordinates, the radius) do this:

It might be useful to have a calculator program to do polar to/from cartesian coordinate conversions.

$$v = \sqrt{v_x^2 + v_y^2}$$

Then for the θ coordinate:

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

There are two equations that you should memorize for describing projectile motion from an angle θ and initial velocity v_0. The first gives the maximum height of the projectile:

$$h = \frac{v_0^2 \sin^2 \theta}{2g} \tag{3.1}$$

And equation 3.2 gives the range, the horizontal distance attained before the object hits its initial height (usually the ground):

$$R = \frac{v_0^2 \sin(2\theta)}{g} \tag{3.2}$$

3.3 Uniform Circular Motion

When a particle moves in a circular with constant speed, it is still accelerated - the velocity vector changes direction. The velocity vector is always tangent to the path of the particle through the particle's position. The acceleration which does this is called *centripetal acceleration*, and this is its formula:

Note that v in this formula means linear speed.

$$a_c = \frac{v^2}{r} \tag{3.3}$$

3.4 Components of Acceleration in Curvilinear Motion

If the velocity of a particle changes in both magnitude and velocity while it is traveling along a curved path, its total acceleration can be written as the sum of acceleration components in the radial and tangential acceleration, that is $a = a_r + a_t$. Tangential acceleration is just speed over time, like this:

$$a_t = \frac{d\,|v|}{dt} \tag{3.4}$$

and, of course, radial acceleration $a_r = a_c = \frac{v^2}{r}$.

3.4.1 Acceleration Unit Vector Notation

e directions of these vectors – cceleration is inward, hence the tion in the total acceleration.

It is convenient to specify the two components using a new kind of unit vector notation. It uses the unit vectors $\hat{r}$ and $\hat{\theta}$. The unit vector $\hat{r}$ is directed radially outward from the center, and the unit vector $\hat{\theta}$ is directed tangent to the circular path in the increasing θ direction (counterclockwise from positive x). So, using circular unit vector notation for acceleration, you could write:

$$a = a_r + a_t = \frac{d\,|v|}{dt}\hat{\theta} - \frac{v^2}{r}\hat{r} \tag{3.5}$$

3.5 Relative Velocity and Acceleration

Consider a booger falling out of your nose, and you falling out of an airplane - the snot is falling at speed s relative to you and you are falling at speed v relative to the airplane. So, the booger is falling at speed $s' = v + s$ relative to the airplane. Simple enough? You should also meditate on a boat crossing a river - think about its velocity:

- relative to the shore
- relative to the moving water in the river
- relative to an idiot who is swimming upstream and making only a little progress

Chapter 4

Forces

Remember when I said that kinematics was limited to, pretty much, observable motion and stuff? Well this is the fun part, where we look at why things move, why some things move in different ways that others, and some of the multi-variable calculus equations you'll use to figure this stuff out. (Just kidding - I'm in a sarcastic mood right now. What you really use are the three basic laws of motion, and you have SIR ISAAC NEWTON to thank for that.)

Dynamics!

4.1 Meet Classical Mechanics (he knows you already)

The purpose of classical mechanics is to examine the connection between the motion of stuff and the forces acting on that stuff. It gives a really good approximation of what is happening for things that are bigger than sub-atomic and slower than light. (Those are the risky business of quantum mechanics and relativistic mechanics.)

A force is a push or a pull. Basically, a force causes something to accelerate (change velocity). If there is more than one force, the object will accelerate only when there is an *unbalanced force*, or when the *net force* is zero.

Forces are vectors. Add them vectorally!

Equilibrium is when an object is not accelerating - that is, its velocity is constant or it is at rest.

Forces can act upon contact, or they can act at a distance through a *field.*

4.2 Newton's First Law

> An object at rest will remain at rest and an object in motion will continue in motion with a constant velocity unless it experiences a

> net external force.

In other words, it is the nature of an object to remain at rest or in motion at a constant velocity, and the only thing that changes this is some net force.

4.2.1 The Inertial Frame

Newton's first law defines what are called *inertial frames of reference*. These are frames of reference in which an object which is subject to no net force moves at a constant velocity; in other words, a frame of reference where the first law works.

4.3 Inertia

I hope that you would be insulted if I actually "reminded" you that mass and weight are not the same.

Inertia is the property of matter to remain at rest or in uniform motion. **Mass** is the quantity that measures inertia. For example, if there are two objects, one of mass 1kg and the other of mass 2kg, and the same force is applied to each, that force will accelerate the 1kg object twice as much as the 2kg object.

4.4 Newton's Second Law

> The acceleration of an object is directly proportional to the resultant force acting on it and inversely proportional to its mass.

So, we have $a = \frac{\sum F}{m}$ and, the equation you should really, really memorize:

$$\sum F = ma \tag{4.1}$$

Equation 4.1 is, of course, for use with vectors, and if you use it, you'll have to execute it for the x, y, and z components.

The SI unit of force is the Newton, and $1N = 1\frac{kg\,m}{s^2}$.

4.4.1 Weight

Weight is a measurement of the force with which bodies are attracted to the earth. It is a very obvious application of Newton's second law, and works like this: weight is the force (which you want to calculate), the mass is that of the body, and the acceleration is that due to gravity. Magically,

$$F = ma$$

becomes:

$$W = mg$$

Since this depends on g, weight varies from say, planet to planet.

4.5 Newton's Third Law

Remember that garbage about every action having an equal and opposite reaction? This is that garbage:

> If two bodies interact, the force exerted by the first upon the second is equal to the opposite of the force exerted by the second upon the first.

Meditate on this and think about how it works. One common way it shows up is with weight. Say you have a $5kg$ block sitting on a table. Its weight pushes down on the table with a force of $(5kg)\left(9.8\frac{m}{s^2}\right) = 49N$, and so the table pushes back up with a force of $49N$ in the opposite direction of the downward force (upward, obviously). The force that the table provides is called the **normal force**, and it is vital to include it in your problems and diagrams. More precisely, the normal force is always perpendicular to the supporting surface, so when a block sits on a ramp, the normal force is diagonal even thought the force of weight is downward - there is a component of the normal countering the weight, and another pushing across the ramp toward the lower end, which may cause the block to slide down (what it is really doing is sliding across and falling - remember 2-d motion?).

Practice, practice, practice these!

4.6 The Force of Friction

Friction is the force that is experienced by an object as it moves (or tries to move) along or through another object or viscous medium. It comes in two flavors: static and kinetic. Each is related to the normal force that the surface an object is moving along provides by a multiple called the coefficient of friction, written μ.

The force of friction is always opposite to the force applied.

4.6.1 Static Friction

Static friction is the force necessary to cause an object at rest to begin motion. It is always equal and opposite the force applied in the direction of (anticipated) motion up to a threshold. This is determined by a property of interaction between the two types of matter that are in contact and is signified in the coefficient of friction. The force that static friction provides:

$$f_s \leq \mu_s N \tag{4.2}$$

4.6.2 Kinetic Friction

Kinetic friction is the force applied to an object while it is moving along a surface. Applying an identical force in the direction of motion will, therefore, keep the object at equilibrium. The force of kinetic friction is:

$$f_k = \mu_k N \tag{4.3}$$

4.7 Analyzing Forces in Problems

When working on a problem involving forces, it is good practice to follow these steps:

1. Draw free-body diagrams for everything involved.
2. Label all forces acting upon the body. Be sure to include friction and normal force as well as weight.
3. Make sure you have accounted for the effects of all three laws of motion.
4. Check if and how various frames of reference effect motion.
5. Note the relationships between bodies and forces (especially tension for those oh-so-popular pulley problems). This will probably lead to your answer.

Chapter 5

Applying Newton's Laws

Newton's laws are great fun to use in *all kinds* of great physics problems, since forces are everywhere. There's zillions all around! Really!

5.1 Uniform Circular Motion

We already know that centripetal acceleration is $\frac{v^2}{r}$ and $\sum F = ma$, we can get centripetal force too!

$$F_c = m\frac{v^2}{r} \quad (5.1)$$

This is the force required to keep an object going at speed v in a circle of radius r.

Note: When solving problems involving this, you often end up having to solve a system of two equations, both involving trigonometry. In that case, the sane way to solve the system is by dividing the two equations - otherwise you'll lose your mind substituting inverse sines and cosines.

5.2 Nonuniform Circular Motion

This is the case where there is tangential acceleration (remember, that's $\frac{dv}{dt}$). So in addition to acceleration in both radial and tangential directions, there is force in radial and tangential directions.

What happens as the body in motion accelerates tangent to the path of motion is that the force required to keep it in the circular path varies: centripetal force

For a body moving in a vertical circle, describing net cetripetal force (often tension): $T = m\frac{v^2}{r} + mg\,\cos\theta$

varies since the speed varies, and (such as weight in the case of a vertical circle where gravity is present) a component of another force adds to or subtracts from the net centripetal force required.

5.3 Motion in Accelerated Frames of Reference

In non-inertial frames of reference, it is necessary to make up various ludicrous forces in order for Newton's second law to work. An example is centrifugal force which pushes you outward in a roller coaster - in the inertial frame, you are experiencing the inward push of centripetal force, but relative to the accelerating roller coaster you are being pushed into your seat by some unknown magic.

Another example is a ball hanging from a string in a train car. When the car accelerates, the ball hangs back at an angle. To the observer in the car, there is a mystery force pulling the ball back, which is countered by a component of tension. But in the inertial frame, it is visible that the acceleration applied to the ball is the same as that applied to the car is applied to the ball inside with the same magnitude, but in the opposite direction, causing the mysterious deflection.

5.4 Motion In the Presence of Resistive Forces

Viscous media exert a force that retards motion, pointed in the opposite direction, which depends on the velocity of the moving body. This is considered with force proportional to v and proportional to v^2.

5.4.1 Force R proportional to v

Note that down is positive for an object sinking in a viscous medium, and R is upward.

$$R = -bv \tag{5.2}$$

$$\sum F = ma = mg - bv$$

$$a = g - \frac{bv}{m}$$

So initially, acceleration is only g. But R reduces acceleration, and eventually equals the weight of the object. At that point, the object has reached *terminal velocity(v_f):*

$$a = 0 = g - \frac{bv_f}{m}$$

$$0 = mg - bv_f$$

$$v_f = \frac{mg}{b}$$

And then there's τ, the time constant, which is how long it takes to get a velocity of $.63v_f$:

$$\tau = \frac{m}{b}$$

And that useful *because:*

$$v = \frac{mg}{b}\left(1 - e^{\frac{-t}{\tau}}\right) = v_f\left(1 - e^{\frac{-t}{\tau}}\right)$$

5.4.2 Force R proportional to v^2

$$R = \frac{1}{2}\zeta\rho A v^2$$

Where ζ is the empirically derived drag coefficient, ρ is air density, and A is the cross-sectional area of the object. But wait, there's more with terminal velocity!

$$\sum F = ma = mg - \frac{1}{2}\zeta\rho A v_f^2 \tag{5.3}$$

$$a = g - \frac{\zeta\rho A v_f^2}{2m}$$

$$v_f = \sqrt{\frac{2mg}{\zeta\rho A}}$$

Whew!

5.5 Nature's Fundamental Forces

gravitation force is the mutual attraction between objects.

electromagnetic force is the attraction or repulsion between two charged particles in motion relative to one another.

strong nuclear force is what holds nucleons together.

weak nuclear force is a short-range nuclear force that tends to produce instability in certain nuclei.

Only the first two are considered in classical mechanics. The last two have been shown to actually be the same thing, called the **electroweak force.**[1]

[1]See the poem on page 2 to see *electroweak* used in a sentence.

Chapter 6

Work and Energy

The concepts of **work** and **energy** can often be applied to mechanical systems, and they usually provide a more simple solution than Newton's laws. Systems can be analyzed according to the various forms of energy they distribute and convert based upon the principle of *conservation of energy.* In this context, only mechanical energy considered.

6.1 Work with a Constant Force

Work(W) is a change in energy. For a constant force(F), it is defined as the product of the component of force in the direction of displacement and the displacement(s). So, for a force directed at angle θ from the direction of positive displacement:

$$W = (F\ \cos\theta)\, s \tag{6.1}$$

The unit of work is the $N \cdot m$ (Newton meter) or the joule (J).

6.2 Vector Dot Products

The scalar product of two vectors, also known as the dot product, is a mathematical figment[1] which is useful in solving problems. In general, the dot product of two vectors we'll call $\overrightarrow{A}$ and $\overrightarrow{B}$ is defined as a scalar quantity equal to the

[1]Vector cross-products, unlike dot products, actually do make sense physically, since the result is a vector and not scalar.

product of the magnitudes of $\overrightarrow{A}$and $\overrightarrow{B}$ and the cosine of the angle θ between their directions:

$$\overrightarrow{A} \bullet \overrightarrow{B} = AB\,\cos\theta \tag{6.2}$$

Remember, $W = \overrightarrow{F} \bullet \overrightarrow{s} = FS\,cos\theta$!

It is also worth noting that the commutative and distribute properties of multiplication hold true for dot products, so $\overrightarrow{A} \bullet \overrightarrow{B} = \overrightarrow{B} \bullet \overrightarrow{A}$ and $\overrightarrow{A} \bullet \left(\overrightarrow{B} + \overrightarrow{C}\right) = \overrightarrow{A} \bullet \overrightarrow{B} + \overrightarrow{A} \bullet \overrightarrow{C}$. There is also some stuff to note about dot products with unit vectors:

$$i \bullet i = j \bullet j = k \bullet k = 1 \tag{6.3}$$

$$i \bullet j = i \bullet k = j \bullet k = 0 \tag{6.4}$$

I think an example is justified, even though typing in all these big dots is a bit annoying:

$$\overrightarrow{A} = A_x i + A_y j + A_z k \qquad \overrightarrow{B} = B_x i + B_y j + B_z k$$

$$\overrightarrow{A} \bullet \overrightarrow{B} = A_x B_x + A_y B_y + A_z B_z$$

6.3 Work with a Varying Force

Calculating work done with forces that vary involves using integrals. This can be represented (in the one-dimension case) by the area under the curve a graph of force vs. displacement.

6.3.1 Working in a Straight Line

As previously stated, work is the area under the curve. Mathematically, that is:

$$W \approx \sum_{x_i}^{x_f} F_x \Delta x$$

Where Δx is a small increment of displacement. As you should know from math class, the limit of this summation as these increments approach 0 (or the number of increments approaches ∞) is:

$$\lim_{\Delta x \to 0} \sum_{x_i}^{x_f} F_x \Delta x = \int_{x_i}^{x_f} F_x dx$$

So, you should memorize that:

$$W = \int_{x_f}^{x_i} F_x dx \tag{6.5}$$

Remember, this only works right for one dimension.

6.4 Work Done by a Spring

Hooke's law states the the force supplied by a spring (where there is no gravity or other external force) is proportional to the opposite of the distance the spring is stretched from its relaxed state by a multiple called the spring **force constant** (unit $\frac{N}{m}$):

$$F_s = -kx \tag{6.6}$$

The force constant is spring-specific, and is larger for stiffer springs. The negative sign signifies that the force of the spring is always opposite the displacement from the equilibrium position. Now, some fun cases:

- Work done by a spring from equilibrium to x_m:

$$W_s = \int_{x_i}^{x_f} F_s \, dx = \int_{-x_m}^{0} (-kx) \, dx = \frac{1}{2}kx_m^2 \tag{6.7}$$

- Work done by a spring over an arbitrary displacement $x_i \to x_f$:

$$W_s = \int_{x_i}^{x_f} (-kx) \, dx = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2 \tag{6.8}$$

- Work of an applied force:

$$W_{F_{app}} = \int_0^{x_m} F_{app} \, dx = \int_0^{x_m} kx \, dx = \frac{1}{2}kx_m^2 \tag{6.9}$$

6.5 The Work-Energy Theorem with Kinetic Energy

Work is a change in energy. We know that kinetic energy is defined as $K \equiv \frac{1}{2}mv^2$, so if a particle of mass m goes from velocity v_i to v_f, then we can determine ΔK, the change in kinetic energy. This change is the work:

$$W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \tag{6.10}$$

This idea eventually builds to a more complete conservation-of-energy approach to analyzing physical systems (and physics problems) which is immensely pleasing. There is more on work-energy in section 6.

6.6 Power

Power is the time rate of energy transfer. So, $P_{avg} \equiv \frac{\Delta W}{\Delta t}$, and more precisely:

$$P \equiv \frac{dW}{dt} \tag{6.11}$$

Since $dw = F \bullet ds$, we can do some useful substitution:

$$P = F \bullet \frac{ds}{dt} = F \bullet v \tag{6.12}$$

The unit of power is the **watt**(W):

$$1W = 1\frac{J}{s} = 1\frac{kg \cdot m^2}{s^3} \tag{6.13}$$

Chapter 7

More with Energy Conservation

This section extends the idea of energy conservation that was brought up in section 6. Using this approach (and other conservation approaches such as momentum), it becomes significantly easier to analyze complex interactions involving physical energy.

7.1 Where it Works

Energy conservation only works when only conservative forces are involved (the one exception being when there's one exception, and you can use conservation to find the difference). A force is conservative if the work it does on a particle moving between two points is independent of the path the particle takes between those points. Additionally, if the particle follows any closed path and returns to its initial position, the net work done by that force is zero.

The work-energy theorem states that the net work done on a particle displaced between two point is equal to its change in kinetic energy.

A force being conservative means that when a problem (er, physical situation) involves that force and it somehow causes a change in some energy, it doesn't "disappear," but rather it turns into a different form of energy that we know about.

7.2 Potential Energy

We know that the work done by a conservative force is only a function of a particle's initial and final coordinates. Potential energy is defined so that work is equal to the decrease in potential energy:

$$W_{F_{conservative}} = \int_{x_i}^{x_f} F_x \, dx = -\Delta U \tag{7.1}$$

You may have noticed from a small detail in equation 7.1 that this applies only for conservative forces.

7.3 Conservation of Mechanical Energy

Imagine this... when work is done by a conservative force on a body in a conservative system, kinetic energy is changed by a certain amount, and potential energy is changed by the opposite of that amount. This is *conservation of mechanical energy*, and it looks like this:

This is a specific instance of the general idea of conservation of energy, which states that energy can neither be created nor destroyed.

$$\Delta K + \Delta U = \Delta (K + U) = 0 \tag{7.2}$$

It's is pretty simple, but that is where its beauty lies.

7.4 Conservation with Nonconservative Forces

If you know the energy roles of all forces but one, and only that one is nonconservative, you can use the process of elimination to learn about how it has subverted the conservative powers that dominate the military-industrial complex at power in the US. No wait, that's conservatism, not conservation. Anyway:

$$W_{nc} = \Delta K + \Delta U = \Delta E$$

7.5 More on Potential Energy

It turns out this potential energy stuff comes in all kinds of cool flavors, like gravity, spring, and razzleberry blue. But first, let's make a quick observation about the relationship between a conservative force and potential energy. First, we know that:

$$W_x = \int F_x \, dx = -\Delta U$$

And therefore:

$$F_x = -\frac{dU}{dx}$$

Nothing monumental, but it's worth noting the relationship.

7.5.1 Gravitational Potential Energy

Well, what do you know, gravity is a conservative force! That means its work grows potential energy:

$$W_g = (mg)\, h = \Delta U_g \tag{7.3}$$

So, something high up has some gravitational potential energy. As it falls, this gets converted into kinetic energy. By the time it hits the ground, all of the gravitational potential energy has been turned into kinetic energy.

7.5.2 Potential Energy in a Spring

Recall that $W_s = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2$. We now add the definition of elastic potential energy for a spring:

$$U_s \equiv \frac{1}{2}kx^2 \tag{7.4}$$

So the total energy of a particle on the end of a spring is this potential energy plus the particle's kinetic energy.

7.6 Stability of Equilibrium

The potential energy curve can often be used to qualitatively analyze the motion of a system. Here is how you can learn about where equilibrium happens and how likely it is to stay that way, just based on the twists and turns of U.

Stable Equilibrium corresponds to minima on $U(x)$.

Unstable Equilibrium corresponds to maxima on $U(x)$.

Neutral Equilibrium happens when $U(x)$ is constant since there are no external forces acting.

Chapter 8

Momentum

Momentum describes interactions between multiple particles. It also fits in with conservation of energy.

8.1 Linear Momentum

The linear momentum of a body of mass m moving at velocity v:

$$p \equiv mv \tag{8.1}$$

The time rate of change of the momentum of a particle equals the net force on the particle:

$$F = \frac{dp}{dt}$$

Change in momentum is called *impulse (I):*

$$I = \Delta p = \int_{t_i}^{t_f} F \, dt = F \cdot \Delta t \tag{8.2}$$

Impulse is a force applied over time: whenever a force is in contact with a particle for a specified time interval, it is imparting momentum.

8.2 Conservation of Momentum

In a system of two particles, momentum is conserved:

$$p = p_1 + p_2 = constant$$

Or, before and after:

$$p_{1_i} + p_{2_i} = p_{1_f} + p_{2_f}$$

8.3 Collisions

In a collision, momentum and/or kinetic energy are/is conserved. Here's the truth table:

Inelastic collision: momentum is conserved, kinetic energy is not.

Elastic collision: both momentum and kinetic energy are conserved.

Perfectly inelastic collision: the objects stick together after the collision, so their velocities are equal: $m_1 v_{1_i} + m_2 v_{2_i} = (m_1 + m_2) v_f$

To analyze collisions in multiple dimensions, use trigonometry, as momentum is a vector and thus has components, which must each be treated separately.

8.4 Center of Mass

Until now, we have talked about particles in motion as one point somehow representing the mass and other characteristics of a whole body. Actually this is a reasonable idea, because any mechanical system actually does move (from the external perspective) as though all its mass was concentrated at one point. That point is called, logically, the center of mass. If a force is applied directly at the center of mass, the whole system will accelerate with acceleration equal to $\frac{F}{m}$, where m is the mass of the entire body.

8.4.1 Calculating the Center of Mass Using Algebra (for bodies of discrete particles)

Algebraically, the center of mass of a system is given by the quotient of the sum of the products of each mass and its center of mass and the sum of the masses. For n mass elements:

$$x_c = \frac{x_1 m_1 + x_2 m_2 \cdots + x_n m_n}{m_1 + m_2 \cdots + m_n} \tag{8.3}$$

This needs to be done along each axis for multiple dimensions, or it may be written as a vector r_c.

8.4.2 Calculating the Center of Mass Using Calculus (for precise calculations on rigid bodies)

If we let the number of mass elements n approach infinity, each element of mass Δm in a body of total mass M:

$$x_c = \lim_{\Delta m_i \to 0} \frac{\sum x \, \Delta m}{M} = \frac{1}{M} \int x \, dm \tag{8.4}$$

8.4.3 Motion of a System of Particles

When a system moves, it can be analyzed as a particle at the center of mass. This holds for forces, velocity, momentum, etc.

Chapter 9

Rotational Motion About a Fixed Axis

When a solid body rotates about an axis, each particle in it (or rather, each ring of particles with the same radius), from the axis outward, has its own velocity and acceleration. Therefore, it is necessary to treat it as such. Luckily, rather than massively transforming the concepts of linear motion so that we can analyze rotation, we define rotational analogues of translational concepts. It turns out these work perfectly, and it's quite astoundingly simple. However, like with translational motion, it takes practice to be able to analyze problems involving rotation.

I found this out the hard way: on the test.

This study of rotation deals with *pure rotational motion*, which is the motion of a rigid body about a fixed axis. Although most real bodies are not rigid, it's no worse an approximation than a spherical bear.[1]

9.1 Rotational Measurement

Imagine a point in a glob rotating in the xy plane. Using polar coordinates to describe its location vector, we say it is at radius r and at an angle from positive x called θ. We describe θ using radians (recall that one radian is the angle subtended by an arc length (s) equal to the radius (r) of the arc: $s = r\theta$, $\theta = \frac{s}{r}$).

There are 2π radians in a circle. To convert to degrees, multiply by $\frac{\pi}{180\,\mathrm{deg}}$.

9.1.1 Velocity

The symbol for angular velocity is ω. Since the radius of each particle in a rotating body is constant, angular velocity is simply the time rate of change in

[1] *See margin note on page 15.*

Note that since θ increases in the counterclockwise direction, positive ω means counterclockwise rotation.

θ, $\frac{\Delta\theta}{\Delta t}$. Or, for instantaneous angular velocity:

$$\omega = \frac{d\theta}{dt} \tag{9.1}$$

9.1.2 Acceleration

The symbol for angular acceleration is α. Naturally, it is the time rate of change in ω, $\frac{\Delta\omega}{\Delta t}$. Or, for instantaneous angular acceleration:

$$\alpha = \frac{d\omega}{dt} \tag{9.2}$$

9.1.3 Direction of the Velocity Vector

The velocity vector in rotational motion is actually directed along the axis of rotation. There is a right-hand-rule to help you remember what its direction is:

1. **Imagine, if you will** that you are hitchhiking. Give the thumbs-up sign.
2. Adjust the world so that, as you thumb your nose at it all, your disinterested extremities[2] are curled along the direction of rotation.
3. The vector ω is directed toward your brain, which is logical since it is the epicenter of all human intellect.

Simple enough? As if *you* would have difficulty grasping it.

9.2 Rotation with Constant ω

Recall translational kinematics equations 2.3, 2.4, 2.5, and 2.6.[3] They have very nice rotational analogues:

$$\omega = \omega_0 + \alpha t \tag{9.3}$$

$$\theta = \theta_0 + \omega_o t + \frac{1}{2}\alpha t^2 \tag{9.4}$$

$$\omega = \sqrt{\omega_0^2 + 2\alpha\left(\theta - \theta_0\right)} \tag{9.5}$$

I personally think that this is one of the most beautiful things to behold – the perfect abstraction of physical formulas is simply striking.

[2]Fingers only, please.

[3]See page 16.

9.3 Rotation to Translation

There some quite nifty equations that show the relationships between rotational and translational motion. They all build from the relationship $s = r\theta$:

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega \tag{9.6}$$

Since motion is in a circle, there are expressions for tangential and radial/centripetal acceleration:

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha \tag{9.7}$$

$$a_r = \frac{v^2}{r} = r\omega^2 \tag{9.8}$$

Also, note that total linear acceleration is $a_t + a_r$. Its magnitude:

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4}$$

9.4 Rotational Kinetic Energy

If you let the subscript i specify a property of a single particle in a rotating body, that particle's kinetic energy would be $K_i = \frac{1}{2}m_i v_i^2$. So the kinetic energy of the entire body would be the sum of the kinetic energies of all these particles, $K = \sum K_i = \sum \frac{1}{2}m_i v_i^2 = \frac{1}{2}\sum m_i r_i^2 \omega^2$, or in the pretty form:

$$K = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

If we compare this to translational kinetic energy, it is clear that $\left(\sum m_i r_i^2\right)$ is the rotational analogue of mass. It is called the *moment of inertia* (I), and with it, kinetic energy is:

$$K = \frac{1}{2}I\omega^2 \tag{9.9}$$

9.5 Calculating the Moment of Inertia

The moment of inertia specifies the same thing as mass does: the ratio of force (we'll get to the rotational analogue of force later) to acceleration ($F = ma$, $m = \frac{F}{a}$.) But, it's a bit more difficult to calculate than mass. First, let's imagine the body as a bunch of volume elements, each with mass Δm. That means $I = \sum r^2 \Delta m$. To be precise:

$$I = \lim_{\Delta m \to 0} \sum r^2 \Delta m = \int r^2 \, dm \tag{9.10}$$

But in order to use this, we need to know the mass element dm in terms of its coordinates. This is described by *mass density* (ρ), or for 3-dimensional bodies *local volume density,* a.k.a. *mass per unit volume*. So:

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta v} = \frac{dm}{dv} \tag{9.11}$$

And thus:

$$dm = \rho \, dv$$

$$I = \int \rho r^2 \, dV$$

Some more tools for work in three dimensions:

Mass per unit area: $\sigma = \rho T$, where T is thickness (of, perhaps, the proverbial pancake.)

Mass per unit length: $\lambda = \rho A$, where A is area.

9.5.1 Parallel Axis Theorem

Given that I_c is the moment of inertia about an axis through the center of mass of a body of total mass M, for any axis parallel and a distance D away from the axis through the center of mass, the moment of inertia about the new axis is:

$$I = I_c + MD^2 \tag{9.12}$$

9.5.2 Perpendicular Axis Theorem

In three dimensions, given that you know the moment of inertia about two perpendicular axes through a point, the moment of inertia about the third perpendicular axis is the sum the two others:

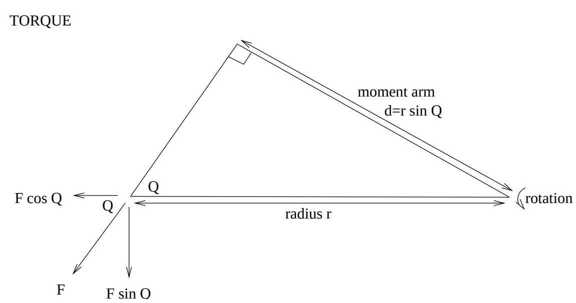
$$I_z = I_x + I_y \tag{9.13}$$

Practice calculating moments of inertia. They are not very easy to comprehend without doing.

9.6 Torque

Torque is sort of like the rotational component of a force. Actually, it is the rotational analogue of force, although it does involve force:

*Excuse the crapulence of this drawing, but I dare you to try using **xfig** for anything serious.*



Torque is the force applied at the distance from the point of rotation:

$$\tau = Fd \tag{9.14}$$

Or, in the figure where the force is applied at angle Q:

$$\tau = rF\,\sin Q$$

An easy was to measure torque is to use to **moment arm** of the force (see the diagram). Torque is the product of this and the magnitude of the force - the length of the moment arm takes care of the angle. Also remember that torque is defined only with regard to a reference axis of rotation.

9.6.1 Relationship Between τ and ω

This part is best expressed entirely in math:

$$F_t = ma_t$$

$$\tau = F_t r = (ma_t)\, r$$

$$a_t = r\alpha$$

$$\tau = (mr\alpha)\, r = \left(mr^2\right) \alpha$$

$$\tau = I\alpha \tag{9.15}$$

There you have it, $F = ma$ for rotation.[4]

9.7 Work and Energy for Rotational Motion

Just as work[5] is the product of force and displacement for translational motion, it is the product of torque and change in θ for rotation:

$$dW = \tau\, d\theta \tag{9.16}$$

And power, the time rate of change in work:

$$P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau\omega \tag{9.17}$$

And for net work, we can also take the change in rotational kinetic energy:

$$W = \frac{1}{2} I\omega^2 - \frac{1}{2} I\omega_0^2 \tag{9.18}$$

[4]See section 4.4 on page 22 for information on Newton's second law.

[5]See section 6 on page 29 for information on translational work and energy.

Chapter 10

Rotational Motion About a Non-fixed Axis

This section deals with the more common case of rotational motion, that is, when the axis is not fixed in space. Only pure rolling motion (no translation/slipping) of homogeneous shapes is discussed.

10.1 Rolling Motion

When a round, homogeneous object rolls, such as a cylinder of mass M and radius R, points of different radii move differently. For the center of mass:

$$v_c = R\omega$$

$$a_c = R\alpha$$

The kinetic energy of a rolling body is the sum of its translational and rotational kinetic energies:

$$K = \frac{1}{2}I_c\omega^2 + \frac{1}{2}Mv_c^2 \tag{10.1}$$

And for pure rolling motion we can substitute in some stuff to get:

$$K = \frac{1}{2}v_c^2\left(\frac{I_c}{R^2} + M\right)$$

This is useful for energy-based analysis of rolling-body situations. If the body is released from rest at the top of a ramp at height h, it has potential energy of Mgh, and thus get final velocity of the center (v_{cf}) when it reaches the bottom of the ramp:

$$U_i = K_f$$

$$Mgh = \frac{1}{2}v_{cf}^2\left(\frac{I_c}{R^2} + M\right)$$

$$v_{cf} = \sqrt{\frac{2gh}{1 + \frac{I_c}{MR^2}}}$$

10.2 The Vector Cross Product

When a force F is applied to a point at position r, the torque caused is defined as their vector product (or cross product):

$$\tau \equiv r \times F \tag{10.2}$$

In general, if $\overrightarrow{C} = \overrightarrow{A} \times \overrightarrow{B}$, the magnitude of C (scalar) is given by:

$$C = \left|\overrightarrow{C}\right| = |AB\,\sin\theta| \tag{10.3}$$

The direction of $\overrightarrow{C}$ is perpendicular to the plane where $\overrightarrow{A}$ and $\overrightarrow{B}$ lie, and its direction is indicated by another right-hand-rule:

1. Give the thumbs-up.
2. Place your hand along $\overrightarrow{A}$ and curl your fingers toward $\overrightarrow{B}$.
3. Your thumb points in the direction of the vector product $\overrightarrow{C}$.

Some interesting properties of cross-products:

$$A \times B = -(B \times A)$$

$$A \times (B + C) = A \times B + A \times C$$

$$\frac{d}{dt}(A \times B) = A \times \frac{dB}{dt} + \frac{dA}{dt} \times B$$

And, behavior with unit vectors:

$$i \times i = j \times j = k \times k = 0$$

$$i \times j = -j \times i = k$$

$$j \times k = -k \times j = i$$

$$k \times i = -i \times k = j$$

Or, in the form you may want to memorize:

$$A \times B = (A_y B_z - A_z B_y)\, i + (A_z B_x - A_x B_z)\, j + (A_x B_y - A_y B_x)\, k \qquad (10.4)$$

10.3 Angular Momentum of a Particle

The *instantaneous angular momentum* (L) of a particle relative to the origin is the vector product of its instantaneous vector position r and instantaneous linear momentum p:

$$L \equiv r \times p \qquad (10.5)$$

L is a vector and you can use the right-hand-rule to get its direction. Its magnitude is:

$$|L| = mvr\, \sin\phi \qquad (10.6)$$

(Where ϕ is the angle between r and p.)

We know that force is the time rate of change of momentum. This is the rotational analogue:

$$\frac{dL}{dt} = r \times F = r \times \frac{dp}{dt}$$

$$\tau = \frac{dL}{dt} \qquad (10.7)$$

10.3.1 Momentum of a System of Particles

Since even distribution of particles in a body makes net internal torque nothing, it is clear that the total angular momentum of a system can vary with time only if there is a net external torque. Thus, the time rate of change of the angular momentum of a system about an origin in an inertial frame equals the net external torque.

10.3.2 With a Rigid Body About a Fixed Axis

For any ith particle, its angular momentum can be expressed as:

$$L_i = m_i r_i^2 \omega$$

So for the entire body, with L along the axis of rotation:

$$L_z = \sum m_i r_i^2 \omega = I\omega \tag{10.8}$$

Which leads again to:

$$\tau = \frac{dL_z}{dt} = I\frac{d\omega}{dt} = I\alpha \tag{10.9}$$

Conservation of Angular Momentum

If the moment of inertia of a body changes (or doesn't), but its angular velocity is not subjected to any torque, its angular momentum is constant:

$$I_i \omega_i = I_f \omega_f$$

Also, the net torque acting on a body about the center of mass equals the time rate of change of angular momentum, regardless of the motion of th e center of mass.

Chapter 11

Static Equilibrium and Elasticity

Equilibrium means that an object is either at rest or moving with constant velocity.

Static Equilibrium deals with bodies at rest.

For particles, rest or movement at a constant velocity is the only condition for equilibrium. However, in real life, not even a spherical bear is a particle, and so there are other conditions: the net force must be zero and the object must have no tendency to rotate – the net torque about any origin must be zero. Therefore, in order to judge whether an object is in equilibrium, we must know its size, shape, and each force acting upon the object as well as its point of application.

This section deals initially with rigid bodies and later with deformable bodies.

11.1 Equilibrium of a Rigid Object

Recall that torque is given by $\tau = r \times F$ (see section 10.2). If the moment arm of a force is nonzero, it produces torque, and unbalanced torque causes acceleration which breaks equilibrium. For a rotating object to be in equilibrium, it must be rotating at a constant angular velocity and not subject to any external torque. In short, for an object to be at equilibrium, the following two conditions must be met:

$$\sum F = 0$$

$$\sum \tau = 0$$

No net force signifies translational equilibrium; no net torque about any origin signifies rotational equilibrium. For *static equilibrium*, an object is at rest and thus has no linear or angular velocity. In other words, forces must balance each other and act along the same line.

If an object is subjected to two forces, the object is in equilibrium only if the two forces are equal in magnitude, opposite in direction, and they have the same line of action (i.e. their moment arms are collinear).

If an object is subjected to three forces, the object is in equilibrium if the forces are *concurrent* – the lines of action of all three forces must intersect at a common point, or they must not intersect (all parallel). This is of course in addition to the required balance of forces (as long as the sum of forces is zero, torque must be zero).

If an object is in translational equilibrium and the net torque is zero about one point, it must be zero about any other point.

11.2 Center of Gravity

When dealing with a rigid object, it is necessary to consider the force of the object's weight. To compute the torque generated by weight, the point of application can be placed at the center of gravity, which coincides with the center of mass when the object is in a uniform field of gravitation. When gravity is not uniform:

$$x_{cg} = \frac{\sum m_i x_i g_i}{\sum m_i g_i}$$

Thus if g is uniform the g_i terms cancel and the equation is the same as for center of mass. Note that any force produced by gravity can be balanced by directing an upward force of the same magnitude as the weight through the center of mass.

11.3 Working Problems with Rigid Bodies in Static Equilibrium

Good problem-solving strategy:

1. Sketch the system.
2. Draw free-body diagrams for all parts showing all external forces.
3. Resolve components for all forces about a reasonably located set of axes or point of rotation. Remember that the placement of these does not affect your solution, try to choose a location that simplifies the problem (this takes practice).
4. Calculate net force and torque, and solve for the conditions of static equilibrium.

Work some sample problems; this type of problem can become quite intuitive after some practice.

11.4 Elastic Properties of Solids

In reality, all objects are deformable to some extent. Welcome to another slice of reality!

The elastic properties of solids are described in terms of *stress* and *strain*.

Stress is the external force per unit cross-sectional area acting on an object.

Strain is a measure of the degree of deformation.

For smallish stresses, it is found that stress is proportional to strain (the constant of proportion depends on the material and type of deformation). This constant is called the *elastic modulus:*

$$Elastic\,modulus \equiv \frac{stress}{strain} \tag{11.1}$$

Young's modulus measures the resistance of a solid to a change in its length.

Shear modulus measures the resistance to motion of the planes of a solid sliding past each other.

Bulk modulus measures the resistance that solids or liquids provide to changes in their volume.

The units of these moduli are force per unit area. (SI: $\frac{N}{m^2}$)

11.4.1 Young's Modulus

Tensile Stress is the ratio of of the magnitude of an external "stretching" force to the crosssectional area of the stretched body, and *tensile strain* is the ratio of stretching length to original length:

$$Y \equiv \frac{tensile\,stress}{tensile\,strain} = \frac{\frac{F}{A}}{\frac{\Delta L}{L_0}} \tag{11.2}$$

This is usually used to describe a rod or wire under tension or compression.

11.4.2 Shear Modulus

Another type of deformation occurs when a body is subjected to a force tangential to one of its faces while friction holds the other face in place (e.g. the top and bottom of a rectangular solid). For a rectangular solid, shear stress tends to produce a shape whose cross-section is a parallelogram. *Shear stress* is the ratio of tangential force to the area (of the face), and *shear strain* is the ratio of the distance the sheared face is displaced to the height of the object (perpendicular distance between faces):

$$S \equiv \frac{shear\,stress}{shear\,strain} = \frac{\frac{F}{A}}{\frac{\Delta x}{h}} \tag{11.3}$$

11.4.3 Bulk Modulus

The bulk modulus deals with the response of an object to uniform squeezing. This means that external forces are acting upon an object at right angles to all of its faces and distributes uniformly over the entire surface area. This quantity (force per area) is called *pressure* and is the *volume stress*. The *volume strain* is the change in volume divided by the original volume:

$$B \equiv \frac{volume\,stress}{volume\,strain} = -\frac{\frac{F}{A}}{\frac{\Delta V}{V}} = -\frac{\Delta P}{\frac{\Delta V}{V}} \tag{11.4}$$

The reciprocal of the bulk modulus is called the ***compressibility*** *of a material.*

This equation bears a negative sign because of the inverse relationship between pressure and volume.

Chapter 12

Oscillatory Motion

A particle experiencing a force which is proportional to position and pulls the particle toward an equilibrium position is said to be in *periodic* or *oscillatory* motion. This happens in, for example, springs and pendula. This section deals with *simple harmonic motion,* where an object oscillates between two positions in space for an indefinite period of time and loses no mechanical energy in the process, whereas in reality retarding forces such as friction are present. When such forces are applied, the oscillations are *damped.* When a force is applied to overcome these damping forces, the object is undergoing *forced oscillation.* (Cases involving damped and forced oscillations are not covered here.)

12.1 Simple Harmonic Motion

A particle in simple harmonic motion has a position according to the following relationship:

$$x = A\cos(\omega t + \delta) \tag{12.1}$$

Where:

A is the **amplitude** of the motion (peak absolute value of x).

ω is the **angular frequency.**

δ is the **phase constant** (or phase angle).

$(\omega t + \delta)$ is called the **phase** of motion and can be used to compare equations in this form.

Note that $x(t)$ is periodic and repeats whenever ωt increases by 2π radians. Thus, the period (T) is given by:

$$T = \frac{2\pi}{\omega} \tag{12.2}$$

The inverse of the period is the *frequency.* It is the number of oscillations per unit time, and its units in SI are $Hz = \frac{1}{s}$.

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \tag{12.3}$$

If we rearrange equation 12.3, we can get:

$$\omega = 2\pi f = \frac{2\pi}{T}$$

ω is the angular frequency from the preceding list. Its units are $\frac{rad}{s}$.

12.1.1 Analysis of Motion

Like any other motion, we can apply calculus to get velocity and acceleration from position:

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \delta) \tag{12.4}$$

$$a = \frac{dv}{dt} = -\omega^2 A \cos(\omega t + \delta) \tag{12.5}$$

But since $x = A \cos(\omega t + \delta)$, substitution turns equation 12.5 into:

$$a = -\omega^2 x \tag{12.6}$$

Extra credit: look at a graph of these functions.

With some more math, we can observe the maximum velocity and acceleration:

$$v_{max} = \omega A \tag{12.7}$$

$$a_{max} = \omega^2 A \tag{12.8}$$

Note that the phase of the velocity differs from the phase of the displacement by $\frac{\pi}{2} radians = 90\,\text{deg}$. The acceleration differs by $\pi\, radians = 180\,\text{deg}$. Thus, when x is zero, speed is maximized; when x is maximized (equal to A) acceleration is a maximum. This of course makes sense qualitatively. Unless you're on a week-long brain fart, which is not uncommon among Physics AP students.

Keep in mind that the solution $x = A\,\cos(\omega t + \delta)$ is a general description of simple harmonic motion, and the phase constant δ and amplitude A are important when comparing two different motions in the same mathematical form. By mixing some of our previous equations and letting x_0 and v_0 be initial position and velocity, respectively, we get:

$$x_0 = A\,\cos\delta$$

$$v_0 = -\omega A\,\sin\delta$$

$$\frac{v_0}{x_0} = -\omega\,\tan\delta$$

$$\tan\delta = -\frac{v_0}{\omega x_0} \tag{12.9}$$

Also, if we solve $x_0^2 + \left(\frac{v_0}{\omega}\right)^2 = A^2\cos^2\delta + A^2\sin^2\delta$ for A, we get:

$$A = \sqrt{x_0^2 + \left(\frac{v_0^2}{\omega}\right)^2} \tag{12.10}$$

In summary, we know δ and A if x_0, ω, and v_0 are given. Some more important characteristic of simple harmonic motion:

For homework, write a pickup like involving "simple harmonic motion."

1. x, v, and a all vary sinusoidally with time but are not in phase.
2. Acceleration increases as displacement increases, but in the opposite direction (imagine a spring).
3. The frequency and period of motion are not dependent upon the amplitude.

12.2 Mass on a Spring

Recall from section 6.4 that a spring lying horizontally on a frictionless surface with a mass on its end will exhibit simple harmonic motion if the mass is moved from equilibrium. Remember that the spring exerts a force proportional to displacement according to Hooke's Law, $F = -kx$. (See page 31 for details.)

This is called a *linear restoring force*, since it is linearly proportional to displacement and always in the direction of equilibrium — opposite the displacement. Throwing in $F = ma$, we get:

$$a = -\frac{k}{m}x \tag{12.11}$$

So, since $a = \frac{d^2x}{dt^2}$:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \tag{12.12}$$

And if we let $\omega^2 \equiv \frac{k}{m}$ (...be defined as...), we can write equation 12.12 as:

$$a = \frac{d^2x}{dt^2} = -\omega^2 x \tag{12.13}$$

And once again, since equations 12.13 and 12.6 are equivalent, we must return to the magic solution of simple harmonic motion. We must be content to admit our mathematical ineptitude and simply accept $x = A\cos(\omega t + \delta)$ as a gift from THE POWERS THAT BE.

This is evidence that whenever there is a linear restoring force proportional and opposite to displacement, a particle undergoes simple harmonic motion.

Keeping in mind that for spring oscillation,

$$\omega^2 = \frac{k}{m} \tag{12.14}$$

we can use previously determined equations for period and frequency:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \tag{12.15}$$

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{12.16}$$

Note that period and frequency depend only on mass and the spring constant.

It is also worth noting that a vertical spring-mass system still undergoes simple harmonic motion. The mathematical reduction is true, but it can be qualitatively justified because gravity is a conservative force and thus causes no loss of energy.

Also think about some special scenarios. When the initial displacement is the maximum displacement, the phase constant is zero $\left(\tan\delta = \frac{-v_0}{\omega x_0}\right)$ and the maximum velocity (see equation 12.7) occurs at $x = 0$.

12.3 The Pendulum

Pendula exhibit simple harmonic motion with a restoring force provided by a component of gravity.

12.3.1 The Simple[st] Pendulum

The 'simple pendulum' is a point mass held in place by a massless string. The tangential component of gravity provides the restoring force:

$$F_t(\theta) = -mg\,\sin\theta = m\frac{d^2s}{dt^2} \tag{12.17}$$

Where θ is the angle from equilibrium (vertical downward) and s is the arc length displacement. If the string is of length L, $s = L\theta$, and this reduces to:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta \tag{12.18}$$

WAIT! In this case, acceleration is proportional to $\sin\theta$, not just θ— it's not simple harmonic motion! But, as per the aforementioned mathematical ineptitude, we use the slick approximation that $\sin\theta \approx \theta$. This is for θ in radians and is pretty close to correct up to angles of about 10 degrees. If you pick up a pendulum to 50, 63, 79, or 90 degrees, you cannot accurately analyze the motion using simple harmonic motion techniques.

Anyway, using this trick, equation 12.18 becomes:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta \tag{12.19}$$

This equation is of the same form as equation 12.13, so it is simple harmonic motion. Thus, we can rewrite the position function as:

$$\theta = \theta_0\,\cos(\omega t + \delta)$$

where θ_0 is the maximum angular displacement — the analogue of amplitude. Thus, we can get the angular frequency and the period of motion:

$$\omega = \sqrt{\frac{g}{L}} \tag{12.20}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \tag{12.21}$$

Again, an interesting observation: the period and frequency of a pendulum depend only on the length of the string and gravity. So in the same gravitational field, two pendula of the same length will oscillate with equal periods, regardless of any difference in masses.

12.3.2 The Physical Pendulum

The physical or compound pendulum is a rigid body suspended from a compound axis that does not pass through its center of mass. When such a system is displaced from equilibrium, it will oscillate. We know that the torque ($\tau = I\alpha$) applied by the weight is the weight (mg) multiplied by the moment arm through which it acts ($d\,\sin\theta$). So:

$$-mgd\,\sin\theta = I\frac{d^2\theta}{dt^2}$$

The negative sign tells us that the torque tends to decrease θ — it is a restoring torque. Using our previous sneaky approximation, for smallish θ:

$$\frac{d^2\theta}{dt^2} = -\left(\frac{mgd}{I}\right)\theta = -\omega^2\theta \tag{12.22}$$

Solving that equation for some more stuff yields:

$$\omega = \sqrt{\frac{mgd}{I}} \tag{12.23}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}} \tag{12.24}$$

Just as you can use oscillation to measure the mass of a body (inertial balance), you can use the measurement of period here to calculate the moment of inertia of a physical pendulum.

12.3.3 The Torsion Pendulum

A torsion pendulum hangs from a wire and rotates in a plane perpendicular to that wire as the wire twists and un-twists. The torque that the wire provides is proportional to the angular displacement:

$$\tau = -\kappa\theta$$

Where κ is the *torsion constant* of the wire. Since $\tau = I\alpha$:

$$\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta \tag{12.25}$$

And, with $\omega = \sqrt{\frac{\kappa}{I}}$:

$$T = 2\pi\sqrt{\frac{I}{\kappa}} \tag{12.26}$$

Note that for a torsional pendulum there is no small angle restriction, simply the elastic limit of the wire.

12.4 Energy of a Simple Harmonic Oscillator

Conservation of energy is intrinsic to simple harmonic motion. Now, let the math speak:

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \delta)$$

For an oscillating spring-and-mass system, $U = \frac{1}{2}kx^2$, so:

$$U = \frac{1}{2}kA^2 \cos^2(\omega t + \delta)$$

Summing these gives the total energy of a spring-mass oscillator:

$$E = K + U = \frac{1}{2}kA^2\left[\sin^2(\omega t + \delta) + \cos^2(\omega t + \delta)\right]$$

But if $\theta = \omega t + \delta$ and $\sin^2\theta + \cos^2\theta = 1$, this reduces to:

$$E = \frac{1}{2}kA^2 \tag{12.27}$$

Also, since $E = K + U = \frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$, we can solve for velocity as a function of displacement from equilibrium:

$$v = \pm\sqrt{\frac{k}{m}\left(A^2 - x^2\right)} = \pm\omega\sqrt{A^2 - x^2} \tag{12.28}$$

Chapter 13

Universal Gravitation

There are lots of charming stories about Newton and gravity. But it's midnight and I have a test on this and the last chapter in about eight and a half hours. This will be Spartan. Newton was a smart guy. I want to be just like him, except alive.

13.1 Newton's Law of Universal Gravitation

Newton published this along with all the other cool stuff in his 1687 *Mathematical Principles of Natural Philosophy.* (Drum roll…)

$$F = G\frac{m_1 m_2}{r^2} \tag{13.1}$$

This gives the force of attraction between to particles of masses m_1 and m_2 separated by distance r. G is a constant:

$$G = 6.672 \cdot 10^{-11}\frac{N \cdot m^2}{kg^2} \tag{13.2}$$

Naturally, gravitation between two particles is directed along the radius. Using earth radius and mass:

$$mg = G\frac{M_e m}{R_e^2} \Rightarrow g = G\frac{M_e}{R_e^2}$$

Generalizing this, we can get the acceleration g produced by any mass M at any distance R from it's center:

$$g = G\frac{M}{R^2} \tag{13.3}$$

13.2 Kepler's Laws

Kepler made some astoundingly simple observations that described the heavens in mathematical terms. I hear it caused quite an uproar.

1. All planets move in elliptical orbits. The sun is located at one of the focal point of the ellipse.
2. The radius vector from the sun to a planet sweeps out equal areas in equal time intervals.
3. The square of the orbital period of any planet is proportional to the cube of the semimajor axis of the elliptical orbit.

Of course, Kepler didn't explain any of this. Newton did that later.

<Author's note: more on Kepler's laws to come.>

13.3 The Gravitational Field

The gravitational field between two masses is an idea supporting gravity's peculiar characteristic of acting at a distance. A force in general is the product of the field strength and the property producing the field. For gravitation, the gravitational force is the product of the gravitational field strength and mass. Thus, the gravitational field strength is the gravitational force divided by mass:

$$g = \frac{F_{grav}}{m} = \frac{Gm}{r^2} \tag{13.4}$$

Obviously, this is what has been called the acceleration of gravity.

13.4 Gravitation and Energy

Energy methods are here too. Ye gods, they spread like blunt at a Doors concert.

13.4.1 Gravitational Potential

Since $\Delta U = U_f - U_i = -\int_{r_i}^{r_f} F(r)$, and $\overrightarrow{F} = -G\frac{mm\prime}{r^2}\hat{r}$ ($\hat{r}$ is the radius unit vector), we have the potential energy function for a gravitational field:

$$U(r) = -G\frac{mm\prime}{r} \tag{13.5}$$

This can be extended to multiple masses, by dealing with each pair of attracting bodies and taking the sum.

13.4.2 Total Energy of an Orbiting Mass

The total energy is the binding energy of the system:

$$E = -\frac{Gmm\prime}{2r} \tag{13.6}$$

13.4.3 Escape Velocity

Escape velocity is the minimum velocity needed to allow an object to reach infinity with a final velocity of 0.

<Some calculations here>

$$v_{esc} = \sqrt{\frac{2GM_e}{R_e}} \tag{13.7}$$

13.5 Gravitational Force Between a Particle and a Spherical Mass

Gravitation is different for a particle in relation to a spherical shell or solid sphere depending on whether it is outside of or within the sphere.

13.5.1 Spherical Shell

Outside the shell, $F = -G\frac{mm\prime}{r^2}$, where r is measured from the center of mass of the shell (center of the sphere).

Inside the shell, $F = 0$.

13.5.2 Solid Sphere

Outside the sphere, $F = -G\frac{mm\prime}{r^2}$, where r is measured from the center of mass of the sphere (center of the sphere).

Inside the sphere, the particle can be considered to be on top of sphere of radius $r\prime$:

$$F\left(r\prime\right)=-G\frac{mm\prime}{r\prime^{2}}$$

Note both that $m\prime$ is still the sphere's total mass and that this is a linear restoring force. This will show up again as Gauss's Law.

Part II

Electricity and Magnetism

Chapter 14

Electric Fields

As you probably do not recall from section 5.5, the electromagnetic force – which exists between two charged particles – is a fundamental force of nature. This force acts through a field called the electric field. The force between two charged particles is described by Coulomb's law. Oh, and welcome to the second half of Physics AP!

14.1 The Electric Charge

You have probably, at some point in your life, perchance, encountered the effects of an object being rubbed – a comb through hair, wool on a balloon, etc. When an object is given the property which makes it behave in that odd way, it is said to be *electrically charged.* If you still don't know what I'm talking about, run in place on a carpet for a few minutes; then find a total stranger. Walk up to them slowly from behind to make sure they don't notice (excellent if they're listening to music on headphones). Then, reach over and touch the back of their ear with your fingertip. ZAP! That's what you get for not being a paranoid suspicious freak like me, jackass.

There are two distinctly different types of electric charges. These were given the arbitrary names *positive* and *negative* by Benjamin Franklin. One may observe that, when comparing objects bearing these different charges, *like charges repel each other and opposite charges attract each other.*

Another important observation is that electric charge is always conserved: charge is not created when two objects are rubbed together, rather, it is transferred. This happens due to the transfer of (negatively charged) electrons from one object to the other. Thus, if two neutral objects are touched and some electrons move to the second from the first, the first will gain a positive charge and the second will gain a negative charge of the same magnitude.

Robert Millikan (from the magical land of Lilliput!) discovered that electric charge is quantized – that is, it occurs as an integral multiple of a fundamental unit of charge. In other words, for charge q:

$$q = Ne$$

Where N is some integer and e is the magnitude of charge of an electron or proton.

Coulomb first measured the force of attraction between charged particles using a torsion balance, observing that force is proportional to the inverse square of the objects' separation $\left(F \propto r^{-2}\right)$. In summary, our observations of electric fields caused by charges and their properties:

- There are two different kinds of charges, called positive and negative. Like charges attract; opposite charges repel each other.
- The force of attraction between two charges varies with the inverse square of their separation.
- Charge is conserved.
- Charge is quantized.

14.2 Insulators and Conductors

Different materials have varying abilities to conduct electrical charge. Those that allow electric charges to move freely are called *conductors,* and those that do not transport charges freely are called *insulators*. Common insulators include glass and rubber – when such materials are rubbed, only the area rubbed may change in charge. Conversely, conductors like copper or tungsten distribute charge over the entire body. Thus, if you hold and rub rods of lucite and copper, the lucite may attract paper because its surface is charged, but the copper will not, since the charge you rub into it will become dispersed throughout your body.

An altogether – well, not *all together,* but sorta together – different type of material is the *semiconductor.* Semiconductors include silicon and germanium, and their electrical properties can be changed by adding various amounts of foreign atoms to the material. These semiconductive properties make possible most of the technology invented in the last twenty years.

14.2.1 Induction

If the earth can be considered an infinite sink for electrical charge, and a conductor is connected to the earth, the object is said to be *grounded.* Building on this idea, we can see how a conductor may become charged by a process called *induction*.

When a (for example) negatively charge rod is brought near a neutral, non-grounded sphere, the electrons in the sphere near the rod will migrate away from the rod. However, if the sphere *is* grounded, some of the electrons will migrate to the earth. Then, when the ground is removed, the sphere retains an *induced* positive charge. When the rod is removed, the positive charge is distributed evenly within the sphere. Throughout, the rod loses none of its charge.

While charging by induction does not require contact, charging by conduction *does neccessitate contact.*

Insulators can undergo a process similar to charging by induction called *polarization.* In this case, the presence of a charged external object causes charges within individual molecules to realign in an insulator. This is why things like combs and balloons can attract neutral object after being rubbed.

14.3 Coulomb's Law

Coulomb's law is based on experiments which verified that the electric force between two particles is proportional to the product of the charges q_1 and q_2 of the two particles and inversely proportional to the square of their separation r. It is attractive if the particles have like charges and repulsive if charges have a different sign.

$$F = k\frac{|q_1|\,|q_2|}{r^2} \tag{14.1}$$

Where k is a constant which depends on the unit used for charge. In SI, this unit is the coulomb (C), which is defined in terms of a unit current called the ampere (A), current being the rate of flow of charge. If the current passing through a wire is $1\,A$, the amount of charge which will pass through any point on that wire in one second is $1\,C$. So, in SI, the coulomb constant is:

$$k = 8.9875 \times 10^9\,\frac{N \cdot m^2}{C^2} \tag{14.2}$$

(k is often approximated as $9.0{\times}10^9$ for simplicity in calculations.) The constant can also be written as:

$$k = \frac{1}{4\pi\epsilon_0}$$

Where ϵ_0 is the *permittivity of free space* and is:

$$\epsilon_0 = 8.8542 \times 10^{-12} \frac{C^2}{N \cdot m^2} \tag{14.3}$$

The smallest unit of charge that you might possibly care about is that of an electron or proton:

$$|e| = 1.60219 \times 10^{-19} C \tag{14.4}$$

Note that electromagnetic forces are still vectors. You can use unit vectors with them like with universal gravitation, and you can still add them.

14.4 The Electric Field

As gravitational field is defined the gravitational force on a test mass divided by that mass $\left(g = \frac{F}{m_0}\right)$, the electrical field vector is defined similarly, involving a test charge:

$$E \equiv \frac{F}{q_0} \tag{14.5}$$

E has SI units newtons per coulomb. Using this definition, it can be said that an electric field exists at a point if the test charge experiences an electrical force at that point. Adding $F = k\frac{qq_0}{r^2}\hat{r}$ into the electric field definition, we can get the electric field as a function of charge and distance:

$$E = k\frac{q}{r^2}\hat{r} \tag{14.6}$$

Again, this is a vector, so fields should be added using components. This vector sum is expressed here:

$$E = k\sum_i \frac{q_i}{r_i^2}\hat{r}_i \tag{14.7}$$

14.5 Electric Field of a Continuous Charge Distribution

The technique described in the previous section can be used to calculate the electric field caused by a point. However, when points of charge are very close together compared to the particle upon which they act, they can be calculated as a continuous system; that is, they are as a total charge evenly distributed through a volume or surface.

To do this, we must first imagine the charge distribution as divided into many small charge elements, each of charge Δq. Then, we use Coulomb's law to calculate the electric field of such an element upon a specific point. Then, as per the superposition principle, we sum up the fields of all the minute charge elements. Recall the electric field of one element is:

$$\Delta E = k\frac{\Delta q}{r^2}\hat{r}$$

Thus, the total electric field generated by i elements is given by:

$$E \approx k\sum_i \frac{\Delta q_i}{r_i^2}\hat{r}_i$$

When we take the limit of this equation as $\Delta q \to 0$, we get:

$$E = k\int \frac{dq}{r^2}\hat{r} \tag{14.8}$$

Note that this is a vector integration and thus must be handled with extreme caution. Also, to make these integrations possible, one uses the concept of charge density:

Charge per unit volume: $\rho \equiv \frac{Q}{V}$, where Q is the charge. ρ has units of $\frac{C}{m^3}$. (For charge distributed throughout a three-dimensional body.)

Surface charge density: $\sigma \equiv \frac{Q}{A}$. σ has units of $\frac{C}{m^2}$. (For charge distributed over a surface of area A.)

Linear charge density: $\lambda \equiv \frac{Q}{\ell}$. λ has units of $\frac{C}{m}$. (For charge distributed along a line of length ℓ.)

14.6 Electric Field Lines

Electric field lines are imagined as lines pointing in the same direction as the electric field vector at any point. These are their properties:

1. The electric field vector E is always *tangent* to the electric field line at any point.

2. The area density of lines through a plane area perpendicular to the field lines is proportional to the strength of the field upon that area. Thus, E is large when the field lines are close together.

When drawing electric field lines, follow these rules:

1. Lines begin at a positive charge and terminate at a negative charge, except when there is an excess of charge, in which case they terminate at infinity.
2. The number of lines drawn must be proportional to the magnitude of the charge.
3. No lines can cross.

Note that these drawings are just visual representation; although the electric field is quantized in charge, it is not quantized in position.

14.7 Motion of Charged Particles in a Uniform Electric Field

The motion of a charged particle in an electric field is akin to projectile motion under gravity. When a particle of charge q is placed in an electric field E, it has an electric force (a la weight) of qE. So:

$$F = qE = ma$$

$$a = \frac{qE}{m} \tag{14.9}$$

Standard kinematics stuff applies:

$$v_x = v_{x_0}$$

$$v_y = at = -\frac{eE}{m}t \tag{14.10}$$

$$x = v_{x_0}t \tag{14.11}$$

$$y = \frac{1}{2}at^2 = -\frac{1}{2}\frac{eE}{m}t^2$$

Be sure to pay attention to the direction of forces when analyzing such problems.

Chapter 15

Gauss's Law

Like Coulomb's law from the previous chapter, Gauss's law is can be used to calculate electric fields. It is based on the concept of *electric flux.*

Finally, the meaning of the word 'flux' *is revealed! Yes, those science fiction movies were wrong.*

15.1 Veni, Vidi, Fluxi

Recall the discussion of electric field lines from section 14.6 — they are lines (curves actually) which depict the paths a charged particle might take when under the firm grasp of the electric field. Well, electric flux (abbreviated Φ) can be described as the number of electric field lines penetrating a surface.

Consider a flat square surface of area A suspended within a constant electric field of magnitude E, perpendicular to the field. The number of electric field lines which penetrate that area is proportional to the product of the area and electric field. So:

$$\Phi = EA \tag{15.1}$$

Multiplying the SI units, we see that the electric flux has units of $\frac{N\,m^2}{C}$.

When the surface in question is not perpendicular to the electric field, we can use trigonometry to adjust the area so that it corresponds to an area perpendicular to the field. If surface $A\prime$ is at an angle of θ from the perpendicular plane area A, its area is equal to $A\ \cos\theta$, and thus:

$$\Phi = EA\ \cos\theta \tag{15.2}$$

Realistically, however, the electric field may be varied over a surface, so these definitions of flux have application only to minute elements of area. Thus, the total electric flux of a body divided into many area elements each of area ΔA is:

$$\sum E_i \Delta A_i$$

Taking the limit of this summation as $\Delta A \to 0$ yields:

$$\oint E\, dA \tag{15.3}$$

(Note that the integral symbol $\oint$ denotes that the limits of the integral should be constructed to encompass the entire surface – this is a "surface integral.") This integral is used to evaluate the net flux through a surface, which means the number of lines leaving the surface minus the number entering. These go in your "fun integrals" list along with electric fields and moments of inertia.

15.2 Gauss's Law

Gauss's law points out an interesting relationship between the net electric flux through a surface and the charge which that surface encompasses. It states the following proportionality[1]:

$$\Phi = \frac{q}{\epsilon_0}$$

This relationship, which is independent of radius and other nonsense, means that the net flux is through a surface is *only* effected by the charge it encloses. By conjecture, if a charge is without[2] a closed surface, it does not cause a net flux through that surface (since as many lines leave as do enter).

15.2.1 The Superposition Principle

The superposition principle states that the electric field due to many charges is the vector sum of their individual electric fields. This concept can also be applied to electric flux - simply substitute the electric field with the vector sum of fields, or charge with net charge.

[1] Check your book for a development of this equation based on the definition of electric flux and a constant electric field whose magnitude is $E = \frac{kq}{r^2}$.

[2] British use of this word, meaning 'outside of.'

15.2.2 Again, that was...

Gauss's law puts all of the above rambling into one compact form:

$$\Phi = \oint E\, dA = \frac{q_{net}}{\epsilon_0}$$

15.2.3 Applying It

When applying Gauss's law to a charge distribution, you should attempt to create a surface whose symmetry is identical to that of the charge distribution. This is usually a sphere or a cylinder. Thus, since the electric field is constant, you can take E out of the integral and are left with the simple product of field and surface area... Excellent!

15.3 Conductors in Electrostatic Equilibrium

When there is no net motion of charge within a conductor, it is said to be in *electrostatic equilibrium.* In that state, it has the following properties:

1. The electric field anywhere within the conductor is zero: the charge may rearrange itself to oppose external fields or will fall into a neutral distribution.
2. Any net charge must be located on the surface (i.e. a Gaussian surface drawn right below the surface can enclose no net charge, so the effective charge must be over it).
3. Just outside a charged conductor, the electric field is perpendicular to the surface and has a magnitude $\frac{\sigma}{\epsilon_0}$ (σ is the charge per unit area at that point).
4. When a conductor has an irregular shape, charge tends to accumulate around the sharp points (=areas where radius of curvature is least).

Chapter 16

Electric Potential

You should recall the concept of potential from the chapter on work and energy. Well, lo and behold, energy methods can be used with electric forces too. The electrostatic force which is quantified in Coulomb's law is conservative, so the stuff it causes can be described in terms of an electric potential.

16.1 Electric Potential

Let's say we put a test charge q_0 in an electrostatic field of magnitude E — this results in an electric force of $q_0 E$. Now, let us consider the work done as the test charge is pulled to the particles generating the field. For an infinitesimal displacement:

$$dW = F\,ds = q_0 E\,ds \tag{16.1}$$

Since work is the opposite of change in potential energy:

$$dU = -F\,ds = -q_0 E\,ds \tag{16.2}$$

Thus, for a displacement of a test charge from points A to B, the change in potential energy is:

$$\Delta U = -q_0 \int_A^B E\,ds \tag{16.3}$$

The *potential difference*, ΔV, between points A and B is defined as the change in potential energy divided by the test charge:

$$\Delta V = \frac{U_B - U_A}{q_0} = -\int_A^B E\,ds \qquad (16.4)$$

Note that potential difference is something different from potential energy. Thus, potential *difference* represents the work per unit charge that an external agent must provide to move a test charge from point A to point B without a change in kinetic energy.

If we choose the potential to be zero for a point at infinity (really, really, super far from the charges producing a field), then equation 16.4 becomes:

$$V_P = -\int_\infty^P E\,ds \qquad (16.5)$$

This gives us a potential difference between the point P and a point infinitely far away.

Potential difference is energy per unit charge, so its units are $\frac{J}{C}$ or volts:

$$1\,V = 1\,\frac{J}{C}$$

Since the potential field is also electric field times distance, we can also relate volts to the SI unit of electric field $\left(\frac{N}{C}\right)$:

$$1\,\frac{N}{C} = 1\,\frac{V}{m}$$

Also used is the *electron volt*, which is the energy that an electron gains when moving through a potential difference of 1 volt:

$$1\,eV = 1V \cdot e = 1\frac{J}{C} \cdot 1.6 \times 10^{-19}C = 1.6 \times 10^{-19}J \qquad (16.6)$$

16.2 Potential Difference in a Uniform Electric Field

In a uniform field a force is conservative — thus the potential difference between and two points in that field is independent of the path taken between those two points; the work done in moving a test charge from point A to point B is the same regardless of route.

For example, imagine a uniform field along the x axis: for two points A and B separated by distance d parallel to the lines of the field:

$$\Delta V = -\int_A^B E\left(\cos 0^\circ\right)\, ds = -\int_A^B E\, ds$$

But E is constant, so:

$$\Delta V = -E\int_A^B ds = -Ed \tag{16.7}$$

Qualitatively, this is negative because B is a point of lower potential than A. Anyway, w can also get the potential energy pretty easily now too:

$$\Delta U = q_0 \Delta V = -q_0 Ed \tag{16.8}$$

This also shows that a positive charge loses energy when it moves in the direction of the field (which of course makes sense, just like everything else in Physics makes perfect elegant beautiful sense). When a positive charge is accelerated in the direction of the electric field, it gains kinetic energy, and thus loses potential.

When dealing with points whose displacement vector is not perpendicular to the field, the vector dot product[1] is used, so:

$$\Delta V = -\int_A^B E \bullet ds = -E \bullet d = Ed\, \cos\theta \tag{16.9}$$

Note that when $\theta = 90^\circ$, this scalar product is zero — all points in a plane perpendicular to a uniform electric field have the same potential.

A surface where a continuous distribution of points has the same potential throughout is called an *equipotential surface*. These surfaces require no work to move a test charge between any two points on that surface.

16.3 Electric Potential from Point Charges

In the case of a point charge, the electric field is directed radially outward. To determine the electric potential at a distance r from the charge, we must recall that the field as a function of distance is given by $E\, ds = k\frac{q}{r^2}\, dr$. Substituting into equation 16.9:

$$\Delta V = -\int E\, ds = -\int k\frac{q}{r^2}\, dr = -kq\int_{r_i}^{r_f} \frac{dr}{r^2} \tag{16.10}$$

Where r_i and r_f are the initial and final distances from the from the point charge. And now, solving this integral:

This is, of course, independen[t] path between the two radii.

$$\Delta V = kq\left(\frac{1}{r_f} - \frac{1}{r_i}\right) \tag{16.11}$$

We can see that since equation 16.11 represents a difference, the potential due to a point charge at any point a distance r from the charge is given by:

$$V = k\frac{q}{r} \tag{16.12}$$

Conveniently, an algebraic sum — not a vector sum (hooray!) — can be used to evaluate the electric potential due to i point charges:

$$V = k\sum_i \frac{q_i}{r_i} \tag{16.13}$$

16.3.1 Potential Energy

Consider the potential energy of a two-particle system. If a particle of charge q_1 has electric potential V_1 at a point, the work that must be supplied to bring another particle of charge q_2 from infinity to that point without acceleration is q_2V_1, so:

$$U = q_2V_1 = k\frac{q_1q_2}{r} \tag{16.14}$$

Where r is the distance between q_1 and the final location of q_2. Note that if the charges are alike, potential energy is positive (since positive work must be done to bring two like charges together) but negative if they are opposite (since negative work is required — the charges attract).

For systems of more than two charged particles, the total potential energy is given by the sum of the potential energy of each pair of charges.

16.4 Electric Potential from Continuous Charge Distributions

If a continuous charge distribution is considered as consisting of lots of small charge elements, then an element of potential would be given by:

[1]See section 6.2 on page 29.

$$dV = k\frac{dq}{r} \tag{16.15}$$

And solving for electric potential:

$$V = k\int \frac{dq}{r} \tag{16.16}$$

If the electric field is known, equation 16.16 can be written as:

$$V = -k\int_{\infty}^{r} E\,ds \tag{16.17}$$

(This form is useful when the electric field can be found easily using Gauss's Law.)

16.5 Electric Potential of a Charged Conductor

Every point on the surface of a charged conductor has the same potential. At each point, the electric field (E) is perpendicular to the displacement (ds), so $E \bullet ds = 0$. Therefore:

$$\Delta V = -\int_{A}^{B} E \bullet ds = 0$$

The surface of any charged conductor in equilibrium is equipotential. Also, since the field within the conductor is 0, the potential *inside* the conductor is constant everywhere and equal to the potential at the surface.

Chapter 17

Capacitance

A capacitor is a device that stores charge, consisting of two conductors separated by an insulator. The material separating the conductors is called a *dielectric.* Many factors effect the *capacitance* of a capacitor.

17.1 Capacitance

Imagine two conductors near each other, each bearing an equal and opposite charge. This combination, called a capacitor, bears a potential difference V which is proportional to the charge. The capacitance is defined as the ratio of the charge on one conductor to the magnitude of the potential difference:

$$C \equiv \frac{Q}{V} \tag{17.1}$$

The unit of capacitance is the farad (F), which, according to the definition of capacitance, is equal to one Coulomb per volt:

$$1F = 1\frac{C}{V} \tag{17.2}$$

Note that a whole farad is a very large capacitance; most capacitors are measured in the microfarads $\left(1\mu F = 10^{-6}F\right)$ or picofarads $\left(1pF = 10^{-12}F\right)$.

Capacitance depends on the geometry of the capacitor: take as an example a sphere. Recall that the sphere's potential is $V = k\frac{Q}{R}$, so:

$$C = \frac{Q}{V} = \frac{Q}{k\frac{Q}{R}} = \frac{R}{k} = 4\pi\epsilon_0 R \tag{17.3}$$

This shows the interesting fact that the capacitance of a sphere is proportional only to its radius and thus independent of charge and voltage.

17.2 Calculating Capacitance

To calculate the capacitance of a capacitor, simply choose a convenient charge Q and calculate voltage, and then use equation 17.1. This is simple if the capacitor's geometry is uncomplicated, and thus probably impossible if the geometry is overcomplicated. Heh.

17.2.1 Parallel-Plate Capacitor

Imagine two parallel plates of equal area A separated by a distance d, each with charge of magnitude Q. First, we need the voltage. If we're using the formula $V = Ed$, we need the electric field, which is given by $E = \frac{\sigma}{\epsilon_0}$. The surface charge density σ is given by $\frac{Q}{A}$, so $E = \frac{Q}{\epsilon_0 A}$ and $V = \frac{Qd}{\epsilon_0 A}$. Again using equation 17.1, we get:

$$C = \frac{Q}{V} = \frac{Q}{\frac{Qd}{\epsilon_0 A}} = \frac{\epsilon_0 A}{d} \tag{17.4}$$

Note that we are assuming the electric field is only directly between the two plates; although there are curves in the field toward the ends of the parallel plates, the effects of that field can be ignored since the distance between the plates is so small.

17.2.2 Cylindrical Capacitor

A cylindrical capacitor consists of a wire of radius a and an outer cylindrical shell of inner radius b. We can find its capacitance for a length l. Again neglecting end effects, we can consider the electric field to be perpendicular to the axis and between the two cylinders. The potential difference ΔV must be calculated first, using the equation $\Delta V = V_b - V_a = -\int_a^b E\, ds$. Using Gauss's law, we can determine the electric field for a conductor with charge per unit length λ:

$$EA = \frac{q}{\epsilon_0}$$

$$E\left(2\pi r l\right) = \frac{\lambda l}{\epsilon_0}$$

$$E = \frac{\lambda}{2\pi r \epsilon_0} = 2k\frac{\lambda}{r}$$

Note that the this calculation is valid since the only the interior wire contributed to the electric field; the outer conductor does not effect the electric field within. Moving right along:

$$\Delta V = -\int_a^b E\,dr = -2k\lambda \int_a^b \frac{dr}{r} = -2k\lambda \ln b - -2k\lambda \ln a = -2k\lambda \ln\left(\frac{b}{a}\right)$$

Again using the definition of capacitance:

$$C = \frac{Q}{V} = \frac{Q}{2k\lambda \ln\left(\frac{b}{a}\right)} = \frac{Q}{2k \ln\left(\frac{b}{a}\right)\frac{Q}{l}} = \frac{l}{2k \ln\left(\frac{b}{a}\right)} \tag{17.5}$$

This shows that the capacitance of this system is proportional to the length as well as the radii of the cylinders.

17.2.3 Another Spherical Capacitor

This spherical capacitor consists of a solid sphere (radius surrounded by a spherical shell. These, of course, have the opposite charges. The electric field outside of the inner sphere is given by $E = k\frac{Q}{r^2}$. The voltage is given by :

$$\Delta V = kQ\left(\frac{1}{b} - \frac{1}{a}\right) = kQ\frac{(b-a)}{ab}$$

To solve the capacitance:

$$C = \frac{Q}{V} = \frac{Q}{kQ\frac{(b-a)}{ab}} = \frac{ab}{k\,(b-a)}$$

17.3 Capacitor Combinations

When two or more capacitors are combined, they produce a net capacitance. Capacitors are called connected *in parallel* if a wire forks, goes through each capacitor, and rejoins. The are connected *in series* when the wire passes across each capacitor in sequence.

For capacitors in parallel, the total capacitance is equal to the sum of each capacitor's capacitance:

$$C_{par} = C_1 + C_2 + C_3 + \cdots \tag{17.6}$$

For capacitors in series, there is a reciprocal relationship:

$$\frac{1}{C_{ser}} = \frac{1}{C_1} + \frac{1}{C_1} + \frac{1}{C_2} + \cdots \tag{17.7}$$

17.4 Energy Stored In a Capacitor

Work is needed to charge a capacitor — this is because the negative and positive charge must be moved to different plates. Although this actually happens from the source of voltage to the plates of the capacitor, the work done is equal to the work required to move a charge from one plate to the other. Recall that the voltage across the capacitor is constant and equal to the source voltage. Also recall that voltage is equal to work per unit charge $\left(V = \frac{dW}{dq}\right)$. So:

$$\int dW = W = \int V\,dq$$

Since by the definition of capacitance, $C = \frac{Q}{V}$, $\frac{1}{C} = \frac{V}{Q}$ and thus $\frac{1}{C}q = V$. Substituting that in:

$$W = \frac{1}{C}\int q\,dq = \frac{q^2}{2C} \tag{17.8}$$

This gives the potential energy stored in the capacitor, so reducing it further:

$$U = \frac{q^2}{2C} = \frac{(VC)^2}{2C} = \frac{1}{2}CV^2 \tag{17.9}$$

This is applicable to all capacitors regardless of geometry.

17.4.1 Energy Density

The volume between the plates of the capacitor can be given by Ad. Thus, we can *calculate* the density of energy stored within the insulator. This the energy density u:

$$U = \frac{1}{2}CV^2 = \frac{1}{2}\left(\frac{\epsilon_0 A}{d}\right)(Ed)^2 = \frac{1}{2}\epsilon_0 Ad\,E^2$$

$$u = \frac{U}{Ad} = \frac{1}{2}\epsilon_0 E^2 \tag{17.10}$$

17.5 Dielectrics

A dielectric is a material used as an insulator within a capacitor. The *dielectric constant* is the ratio of the capacitance with the new insulator to the capacitance with air:

$$\kappa = \frac{C}{C_0} \tag{17.11}$$

Also, since voltage is directly inversely proportional to capacitance, the capacitance goes up if the voltage decreases while the charge stays fixed. Thus, the dielectric constant can also be written as:

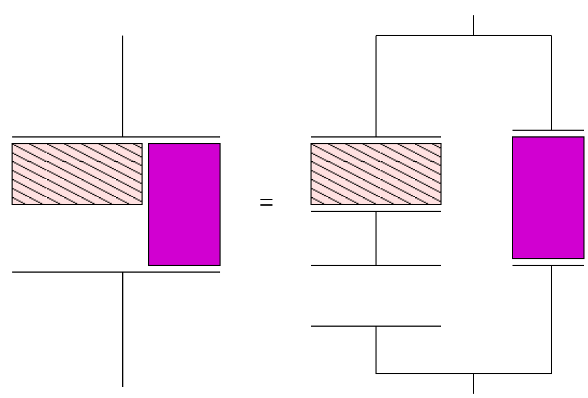
$$\kappa = \frac{V_0}{V} \tag{17.12}$$

As an mnemonic, you can consider the dielectric constant as modifying the permittivity of free space to the permittivity of the dielectric:

$$\epsilon = \kappa\epsilon_0 \tag{17.13}$$

Dielectric strength is the maximum electric field that a dielectric can take before ionizing and breaking down (=becoming a straight conductor). Its units are volts per meter.

A capacitor only partially filled with varying dielectrics can be considered a series of capacitors in parallel or in series:



Chapter 18

Current and Resistance

All previous discussion of electricity has been of charge at rest (electrostatics). This chapter deals with the concept of moving charge, or *current.*

18.1 Electric Current

Current is the movement of electric charges of like sign. Relative to, for example, a crossectional area of a wire, current is the rate at which charge flows through the area:

$$I_{avg} = \frac{\Delta Q}{\Delta t} \tag{18.1}$$

Of course, there is also instantaneous current:

$$I \equiv \frac{dQ}{dt} \tag{18.2}$$

The unit of current in SI is the ampere:

$$1A = 1\frac{C}{s} \tag{18.3}$$

Although usually the actual movement is that of electrons, current is conventionally considered in terms of the flow of positive charge. Moving charges are called *mobile charge carriers.*

Now imagine a section of wire of crosssectional area A and length l. If n is the the number of mobile charge carriers per unit volume, the charge in that volume is given by:

$$\Delta Q = nA\,\Delta x\,q$$

Where q is the charge on each particle. If the particles are moving at a speed v_d, then they move a distance $\Delta x = v_d \Delta t$, so:

$$\Delta Q = nA\,v_d \Delta t\,q$$

But wait... if we divide this equation by change in time, we get:

$$\frac{\Delta Q}{\Delta t} = I = nqv_d A \tag{18.4}$$

The velocity v_d is actually an average velocity, and is called the *drift velocity.* To picture this movement, imagine free electrons within a conductor as molecules in a gas, undergoing effectively random motion. When a potential difference is applied over the conductor, a field is created and causes a force on the electrons which causes them to move and thus generate current. Although in reality the electrons suffer numerous collisions with the molecules of the conductor and thus follow a zigzagging path, they generally move in the opposite direction of E with the average velocity v_d.

18.2 Resistance

Although when a conductor is in electrostatic equilibrium, it can have no electric field within it, an electric field can be present in a non-electrostatic situation where the field induces motion and current.

18.2.1 Current Density

The *current density* within a conductor is the current per area:

$$J \equiv \frac{I}{A} = nqv_d \tag{18.5}$$

The units of current density in SI are $\frac{A}{m^2}$. Also, current density is a vector quantity, in the direction of motion of positive charge.

Current density and electric field occur within a conductor when a potential difference is maintained across it. With a constant potential difference, the current will also be constant. In many cases, the current density is proportional to the electric field within a conductor:

$$J = \sigma E \tag{18.6}$$

Where σ is the *conductivity* of the conductor (not the surface charge density).

18.2.2 Ohm's Law

Materials that follow the relationship given in equation 18.6 follow Ohm's law:

For many materials (most metals), the ratio of current density to electric field is a constant which is independent of the electric field producing the current. The materials that follow Ohm's law are called *ohmic*, and those that do not are called *nonohmic* (tough, I know). Ohm's law is an observed relationship, not a natural law.

But how is this useful? Consider a segment of wire with crosssectional area A and length l. A potential difference ΔV is maintained across the wire from end to end. If we assume that the electric field within the wire is uniform the voltage is equal to the product of field and distance ($V = El$). Thus:

$$J = \sigma \frac{V}{l}$$

Also noting that $J = \frac{I}{A}$, the voltage can be rewritten:

$$V = \frac{l}{\sigma} J = \left(\frac{l}{\sigma A} \right) I$$

The quantity $\frac{l}{\sigma A}$ is called the *resistance* R of the conductor:

$$R = \frac{l}{\sigma A} = \frac{V}{I} \tag{18.7}$$

The unit of resistance is the ohm (Ω), where $1\Omega \equiv 1\frac{V}{A}$. Thus, if a voltage of $1V$ causes a current of $1A$, the conductor has a resistance of 1Ω.

The inverse of conductivity is called the resistivity of a material, and has units of ohm-meters (Ωm):

$$\rho \equiv \frac{1}{\sigma} \tag{18.8}$$

Substituting this definition into equation 18.7 yields:

$$R = \rho \frac{l}{A} \tag{18.9}$$

This shows that in a wire, resistance increases with length and is reduced with increasing thickness of the wire.

18.3 Resistivity of Various Conductors

Among other things, temperature can effect the resistivity of a conductor. For most conductors, the change in resistivity can be calculating using the formula:

$$\rho = \rho_0 \left(1 + \alpha \left(T - T_0\right)\right) \tag{18.10}$$

Where ρ_0 is the resistivity of the material at some reference temperature T_0 and α is the temperature coefficient of resistivity. ρ and T represent current values of resistivity and temperature. Solving equation 18.10 for α, we get:

$$\alpha = \frac{1}{\rho_0} \frac{\Delta\rho}{\Delta T} \tag{18.11}$$

Where $\Delta\rho = \rho - \rho_0$ and $\Delta T = T - T_0$. Also, since resistance is directly proportional to resistivity, the resistivities in equation 18.10 could be replaced with resistances:

$$R = R_0 \left(1 + \alpha \left(T - T_0\right)\right) \tag{18.12}$$

This relationship is often used not to calculate resistances, but to calculate temperatures with a high degree of precision.

18.4 Electrical Energy and Power

Naturally, the concepts of energy and power are quite applicable to electricity — in fact, you may have noticed electricity supplying power to all sorts of useful machines. When a battery established a current in a conductor, the chemical energy in the battery is being converted into kinetic energy propelling the charge carriers. This kinetic energy is lost in collisions between the charge carriers and the lattice ions, causing the conductor's temperature to increase. Thus, in general, the battery's chemical energy is continuously converted into thermal energy.

Anyway, the change in potential energy as a charge Q moves across a potential difference ΔV is the product:

$$\Delta U = Q\Delta V \tag{18.13}$$

Or, if the charge changes:

$$\Delta U = \Delta Q V \tag{18.14}$$

Thus, the time rate of change of potential energy:

$$\frac{\Delta U}{\Delta t} = \frac{\Delta Q}{\Delta t} V = IV \tag{18.15}$$

This is, of course, power:

$$P = IV = I^2 R = \frac{V^2}{R} \tag{18.16}$$

Recall that the unit of power is the watt. The forms of this equations have different applications:

The form $P = IV$ is very generalized... it can apply to pure resistances as well as complex devices like electric motors.

The form $P = I^2 R$ is usually used to calculate energy lost as heat as current flows through a resistance; this is often called "$I^2 R$ loss."

18.4.1 Energy Conversions in Household Circuits

Many household devices use the heat generated by electric current undergoing resistance to function (such as irons, dryers, heaters, jacuzzis, etc.). These devices use energy at a time rate which is called power, as you have seen. What they consume (and what you get billed for) is energy, and generally the preferred unit of energy for electric companies is the kilowatt-hour:

$$1 kWh = 3.6 \times 10^6 J \tag{18.17}$$

Chapter 19

Direct Current Circuits

A circuit contains elements such as batteries, resistors, and capacitors in various combinations. Numerous ridiculously confounding and purposely maligned formulae "help" you to figure out what exactly the hell is going on.

19.1 Electromotive Force

The preceding chapter showed that current can be maintained within a closed circuit through the use of a source of energy called an electromotive force (**emf**, ε). Any device that increases the potential energy of charges circulating in a circuit is a source of emf. A source of emf can be considered a "charge pump" that causes electrons to move in the direction opposite the electrostatic field within the source. Thus, ε describes the work done per unit charge and has units of volts.

The maximum voltage that a source of voltage can produce is called the electromotive force or **emf**. It is sometimes called the *open circuit voltage* since it is measured when no current is flowing. This is because when current flows, internal resistance within the source of voltage causes some of the energy produced to be lost as heat, reducing the source's voltage to $emf - Ir$, where r is the internal resistance. The decrease in voltage due to a current using energy to move through part of a circuit is often called the *voltage drop* across that part.

$$V = \varepsilon - Ir \tag{19.1}$$

Dropping in $V = IR$:

$$IR = \varepsilon - Ir$$

$$\varepsilon = IR + Ir \tag{19.2}$$

Solving for current yields:

$$I = \frac{\varepsilon}{R + r} \tag{19.3}$$

19.2 Resistor Combinations

19.2.1 Resistors in Series

When resistors are connected in series, the same current flows through each resistor, and the voltage is divided among them: $V = IR_1 + IR_2 + \cdots$, where each IR term represents a voltage drop across a resistor. Anyway, what follows is:

$$R_{ser} = R_1 + R_2 + \cdots \tag{19.4}$$

19.2.2 Resistors in Parallel

Resistors in parallel are independent paths for current flow. Thus, each resistance is across the total voltage and $I_i = \frac{V}{R_i}$. Conservation of electric charge mandates that $I = \sum \frac{V}{R_i}$. Thus, $I = \frac{V}{R_1} + \frac{V}{R_2} + \cdots$, and:

$$\frac{1}{R_{par}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots \tag{19.5}$$

19.3 Kirchhoff's Rules

Simple circuits can be analyzed using Ohm's law and resistor combinatronics. However, it is not always possible to reduce a circuit to a single path. In order to analyze more complicated circuits, we employ Kirchhoff's rules:

1. The sum of currents entering any junction (= a point in the circuit where a current can split) must equal the sum of the currents leaving that junction.
2. The algebraic sum of the changes in potential across all of the elements around any closed circuit loop must be zero.

The first rule is simple conservation of charge: whatever charge enters a point in a circuit will leave that point.

The second rule comes from conservation of energy: any charge that moves fully around a closed loop in a circuit, finishing and starting at the same point, gains as much energy as it loses. The following are helpful in calculations:

- If the resistor is traversed in the same direction as the current, the change in potential is $-IR$.
- If the resistor is traversed in the direction opposite the current, the change in potential is $+IR$.
- If the source of emf is traversed in the direction of emf (from negative to positive), the change in potential is $+\varepsilon$.
- If the source of emf is traversed in the direction opposite the emf (from positive to negative), the change in potential is $-\varepsilon$.

The first rule can be applied as many times as there are junctions in the circuit and no more. The second rule can be used as long as a new circuit element is included in each calculation. After you get all the equations together, you take the bigass system of equations and solve it. Hopefully you know something about linear algebra, or at least how to punch this system into your graphing calculator to have it solve it.

The key to using these rules is to know that the direction you assign to currents is mathematically irrelevant; where you assigned the wrong direction, you will end up with a negative current. Thus, assign directions in whatever way seems most convenient for the application of the two rules. Then, apply the first rule to all junctions and the second rule to all loops and solve the system.

19.4 RC Circuits

So far, we have dealt with circuits with constant currents, called *steady-state circuits.* Now, we consider circuits with capacitors, where the currents may change in time. When a voltage is applied to a capacitor, the rate at which it becomes charged depends on its capacitance and on the resistance in the circuit.

19.4.1 Charging a Capacitor

Consider a circuit containing a battery, a resistor, and a capacitor. Between the battery and resistor is a switch. When the switch is charged, charge begins to flow and creates a current within the circuit. The capacitor will begin to charge, with one sign charge building on one side of the capacitor and the other sign traveling through the resistor to the other side. Once the maximum charge on the capacitor is reached (this depends on the emf of the battery) the current in the circuit is zero. Applying Kirchhoff's second rule to the circuit after the switch is closed:

$$0 = \varepsilon - IR - \frac{q}{C} \tag{19.6}$$

Where IR is the potential difference across the resistor and $\frac{q}{C}$ is the potential difference across the capacitor. These are instantaneous values of charge and current as the capacitor charges.

Since the capacitance is zero at the instant that the switch is closed, we can solve equation 19.6 for the initial current:

$$I_0 = \frac{\varepsilon}{R} \tag{19.7}$$

At the instant that the switch is closed, the potential drop across the circuit is only at the resistor. But when the capacitor is charged, the current is zero and the potential drop is only across the capacitor. Throwing this fact into equation 19.6, we can solve for the maximum charge:

$$Q = \varepsilon C \tag{19.8}$$

But what if we want to find the relationship between these properties and time? First, differentiate equation 19.6 with respect to time, noting that ε is constant:

$$\frac{d}{dt}\left(\varepsilon - \frac{q}{C} - IR\right) = 0 - \frac{dq}{C\,dt} - R\frac{dI}{dt} = 0$$

Since $\frac{dq}{dt}$ is current, this can be reiterated as:

$$\frac{I}{C} + R\frac{dI}{dt} = 0$$

$$\frac{dI}{I} = -\frac{dt}{RC} \tag{19.9}$$

Now we can solve for some stuff, noting that at $t = t_0$, $I = I_0$ and R and C are constants:

$$\int_{I_0}^{I} \frac{dI}{I} = \frac{-1}{RC}\int_0^t dt$$

$$\ln\left(\frac{I}{I_0}\right) = -\frac{t}{RC}$$

$$I = I_0 e^{\left(\frac{-t}{RC}\right)} = \frac{\varepsilon}{R} e^{\left(\frac{-t}{RC}\right)} \tag{19.10}$$

We can also solve for charge on the capacitor by replacing I with $\frac{dq}{dt}$:

$$dq = \frac{\varepsilon}{R} e^{\left(\frac{-t}{RC}\right)} dt$$

$$\int_0^q dq = \frac{\varepsilon}{R} \int_0^t e^{\left(\frac{-t}{RC}\right)} dt$$

$$q = C\varepsilon\left(1 - e^{\left(\frac{-t}{RC}\right)}\right) = Q\left(1 - e^{\left(\frac{-t}{RC}\right)}\right) \tag{19.11}$$

19.4.2 Discharging a Capacitor

Now, consider another circuit featuring a capacitor of initial charge Q, a switch, and a resistor. When the switch opens the circuit, there is no current through the circuit, resulting in a potential drop across the resistor of zero, and across the capacitor of $\frac{Q}{C}$. However, when the circuit is closed, the capacitor discharges through the resistor. For a given contemporaneous current I and capacitor charge q, the potential difference must be:

$$IR = \frac{q}{C} \tag{19.12}$$

Here, the current is the rate of decrease in the charge of the capacitor. So:

$$R\left(-\frac{dq}{dt}\right) = \frac{q}{C}$$

$$\frac{dq}{q} = -\frac{dt}{RC}$$

Since the initial charge is Q:

$$\int_Q^q \frac{dq}{q} = -\frac{1}{RC}\int_0^t dt$$

$$\ln\left(\frac{q}{Q}\right) = -\frac{t}{RC}$$

$$q = Qe^{\left(\frac{-t}{RC}\right)} \tag{19.13}$$

This can also be solved for current by differentiation:

$$I = -\frac{dq}{dt} = \frac{Q}{RC}e^{\left(\frac{-t}{RC}\right)} = I_0 e^{\left(\frac{-t}{RC}\right)} \tag{19.14}$$

Chapter 20

Magnetic Fields

Magnets have an important role in making electricity a usable means of energy distribution. The magnetic field generated by a magnet flows from two *poles,* called the *north* and *south* poles; every magnet has these two poles. The field travels from the north to the south pole, opposite poles attract each other, and like poles repel.

20.1 The Magnetic Field

Just as the electric field is defined at a point as the electric force per unit charge acting at that point, and the gravitational field as force per unit mass, the magnetic field is defined in terms of magnetic force exerted upon a charged particle.

The magnetic field vector B, also called the *magnetic induction* or *magnetic flux density*) is the force that will act upon a charged particle moving with velocity v. The magnetic force has the following properties:

1. It is proportional to the charge and speed of the particle.
2. It depends upon the particle's velocity and the magnitude and direction of the magnetic field.
3. When the velocity is parallel to the magnetic field, the force on the moving charge is zero.
4. When there is an angle between v and B, F is perpendicular to the plane of v and B.
5. The magnetic force on a negative charge is opposite that of a positive charge.

6. When θ is the angle between v and B, the magnitude of the force is proportional to $\sin\theta$.

The mathematical expression of all of this is:

$$F = qv \times B \tag{20.1}$$

Which is a vector cross product (see 10.2). Recall that as such, its magnitude is given by:

$$F = qvB\sin\theta$$

And its direction can be determined using a right-hand rule:

1. Point the fingers of your right hand in the direction of v.
2. Curl your fingers toward B.
3. Your thumb points in the direction of the cross product $v \times B$.

This will be the direction of magnetic force upon a positively charged particle; take the opposite or use your left hand for a negative particle.

20.1.1 Differences between the electric and magnetic fields

There are some interesting differences between the electric and magnetic fields:

1. The electric force is parallel to the electric field; the magnetic force is perpendicular to the magnetic field.
2. The electric force is independent of the particle's velocity.
3. The electric force does work in displacing a charge. Since magnetic force is always perpendicular to velocity, it provides a centripetal force which is nonconservative and thus cannot alter the kinetic energy of a particle it effects: $dW = F \bullet ds = (F \bullet v)\ dt = 0$, since $F \perp v$. Thus, an applied magnetic field can alter the direction of a particle's velocity, but cannot change its speed.

The SI unit for magnetic field is the *tesla*, T, or *weber per square meter*, $\frac{Wb}{m^2}$. A unit breakdown:

$$T = \frac{Wb}{m^2} = \frac{N}{C\,\frac{m}{s}} = \frac{N}{A\,m}$$

The cgs unit for magnetic field is the *gauss*, G, which can be related to the tesla by the relationship $1T = 10^4 G$.

20.2 Magnetic Force on a Conducting Wire

We know from equation 20.1 the magnetic force upon a single moving charge. Relating this to the force upon a conductor is relatively simple in purely mathematical form: imagine charges moving through a wire of length ℓ at a constant velocity which is our familiar v. Thus a charge takes some time t to travel through the wire. Substituting some relationships into $F = qv \times B$:

$$F = q\left(\frac{\ell}{t}\right) \times B$$

$$F = \frac{q}{t}\ell \times B$$

Recall that charge per time is current, so:

$$F = I\ell \times B \tag{20.2}$$

20.2.1 Wires of Other Odd Shapes

For any shape of wire conducting a constant current in a constant magnetic field, the force on a teensy segment of wire ds is $dF = I\,ds \times B$. To get the total force, integrate over the length of the wire:

$$F = I\left(\int_a^b ds\right) \times B \tag{20.3}$$

Where a and b are the endpoints of the wire.

Curved Wire

A curved wire in a uniform magnetic field produces various magnitudes and directions of magnetic force at different points along the curve. However, if B is constant and taken out of the integral expression as in equation 20.3, the integral part, $\int_a^b ds$, is simply a vector sum of displacement, which thus depends only on the endpoints. Therefore, a curved wire can be evaluated as if it were a straight wire connecting its endpoints (with $F = I\ell \times B$).

Closed Loop

Here again, we must consider the vector sum, which in this case is zero — since the endpoints, so to speak, are at the same location. Mathematically, the limits on our integral are equal, so there is no force exerted upon a loop of wire in a constant magnetic field. However, the *torque* on such a loop may not be zero. . .

20.3 Torque on a Closed Current-carrying Loop in a Constant Magnetic Field

Imagine a rectangle of sides length a and b in the presence of a magnetic field $B \perp a$. Since $B \| a$, the only force is on the sides of length b, which has a magnitude of $F = I\ell \times B = IbB$. However, since the current in each side of length b is opposite the other, the net force is zero; but clearly there is torque if you assume that the rectangular current loop is free to rotate about an axis parallel to the sides of length b. Assume the axis is through the midpoints of the a sides:

$$\tau = F_b \left(\frac{a}{2}\right) + F_b \left(\frac{a}{2}\right) = F_b a = (IbB)\, a = IabB$$

But since the product ab is otherwise known as the *area(A)* of the current loop, we can write:

$$\tau = IAB \tag{20.4}$$

Since our expressions for force depend on a cross-product, the relationship carries through to be applied when B is not perpendicular to a side of the loop:

$$\tau = IA \times B = IAB\, \sin\theta \tag{20.5}$$

The product IA is called the magnetic moment (μ). It is directed perpendicular to the surface (as the area vector) and its direction depends on the direction of current. Curl the fingers of your right hand in the direction of current and your thumb will point in the direction of μ.

20.4 Motion of a Charged Particle in a Magnetic Field

Since the magnetic (Lorentz) force on a particle is centripetal and effects direction and not speed, it is a centripetal force. Assuming that the particle has some mass (as far as you know, it pretty much has to have mass in order to *be* a something):

$$F = qvB = \frac{mv^2}{r}$$

Thus, the particle travels in a circle with radius:

$$r = \frac{mv}{qB}$$

And angular velocity, frequency, and period:

$$\omega = \frac{v}{r} = \frac{qB}{m}$$

$$f = \frac{\omega}{2\pi} = \frac{qB}{2\pi m}$$

$$T = \frac{1}{f} = \frac{2\pi m}{qB}$$

20.4.1 Mass Spectrometer

In a mass spectrometer, uniformly charged particles are first accelerated through a potential difference, giving them kinetic energy of qV. Then, they pass through a "velocity selector," a region where both an electric and a magnetic field are present and set so that all charged particles not moving at a certain velocity are deflected away from the opening to the final region, where only a magnetic field is present and given the known charge and velocity, the mass of particles can be calculated after measuring the radius of path that they travel.

Chapter 21

Sources of the Magnetic Field

21.1 The Biot-Savart Law

$$dB = k_m \frac{I\,ds \times \hat{r}}{r^2} \tag{21.1}$$

Where $k_m = 10^{-7} \frac{Wb}{A\,m} = \frac{\mu_0}{4\pi}$, $\mu_0 = 4\pi \times 10^{-7}$.

21.2 Magnetic Force Between Two Conducting Wires

If the currents are in the same direction, this attracts. In opposite directions, it repels.

$$\frac{F}{\ell} = \frac{\mu_0 I_1 I_2}{2\pi a} \tag{21.2}$$

Where a is the separation between the wires.

21.3 Ampere's Law

$$\oint B\,ds = \mu_0 I \tag{21.3}$$

21.4 Magnetic Field of a Solenoid

$$\oint B\,ds = B\ell = \mu_0 N I \tag{21.4}$$

Where N is the number of turns of wire around the solenoid. Thus, substituting $n = \frac{N}{\ell}$, the number of turns per unit length:

$$F = \frac{\mu_0 N I}{\ell} = \mu_0 n I \tag{21.5}$$

21.5 Magnetic Flux

$$\Phi_m = \int B\,dA \tag{21.6}$$

21.6 Gauss's Law in Magnetism

Since we are assuming that no magnetic monopoles exist, and magnetic field line that enters a closed surface must exit it. Thus, through any closed surface:

$$\oint B\,dA = 0 \tag{21.7}$$

21.7 Displacement Current

$$I_d = \varepsilon_0 \frac{d\Phi_e}{dt} = \frac{dQ}{dt} \tag{21.8}$$

$$\oint B\,ds = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_e}{dt} \tag{21.9}$$

Chapter 22

Faraday's Law

In 1831, experiments carried out in England by Michael Faraday and independently in the United States by Joseph Henry showed that an electric current can be induced in a wire by a changing magnetic field. This discovery led to Faraday's law of induction.

22.1 Faraday's Law of Induction

The emf induced in a circuit is directly proportional to the time rate of change of magnetic flux through the circuit.

$$\varepsilon = -\frac{d\Phi_m}{dt} \tag{22.1}$$

If there are N loops of wire each with the same area A, the flux is $BA\,\cos\theta$ so:

$$\varepsilon = -N\frac{d}{dt}\left(BA\,\cos\theta\right) \tag{22.2}$$

22.2 Motional EMF

Motional emf deals with the emf induced in a conductor moving through a magnetic field. For a conductor linking two sides of a rectangular circuit, of length ℓ:

$$\varepsilon = -\frac{d\Phi_m}{dt} = -\frac{d}{dt}\left(B\ell x\right) = -B\ell\frac{dx}{dt} = -B\ell v \tag{22.3}$$

22.3 Lenz's Law

The polarity of the induced emf is such that it tends to produce a current that will produce a change in magnetic flux opposing the existing magnetic flux through the loop.

22.4 Maxwell's Equations

The following four equations, collectively known as Maxwell's equations after James Clerk Maxwell, are the fundamental equations of electromagnetism. Enjoy a tasty set of equations applied to free space.

$$\oint E \cdot dA = \frac{Q}{\varepsilon_0} \tag{22.4}$$

$$\oint B \cdot dA = 0 \tag{22.5}$$

$$\oint E \cdot ds = -\frac{d\Phi_m}{dt} \tag{22.6}$$

$$\oint B \cdot ds = \mu_0 I + \varepsilon_0 \mu_0 \frac{d\Phi_e}{dt} \tag{22.7}$$

Chapter 23

Inductance

23.1 Self-inductance

For the toroidal coil or ideal solenoid of N turns:

$$\varepsilon = -N\frac{d\Phi_m}{dt} = -L\frac{dI}{dt} \tag{23.1}$$

Where L is the inductance, given by:

$$L = \frac{N\Phi_m}{I} \tag{23.2}$$

23.2 Circuits with Inductors

A circuit with a coil such as a solenoid has a self-inductance that prevents the current from changing instantaneously. An element of a circuit with a large inductance, called an *inductor*, is represented in a schematic diagram by a series of curls. The current in such a circuit is:

$$I\left(t\right) = I_0 e^{\frac{-t}{\tau}} \tag{23.3}$$

Where $I_0 = \frac{\varepsilon}{R}$ and $\tau = \frac{L}{R}$.

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